Transmission properties of material with relative permittivity and permeability close to -1

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ABSTRACT

J.B. Pendry has shown that a layer of material with relative permittivity and relative permeability both equal to -1 behaves as a perfect two-dimensional lens for an object closer than the thickness of the layer. We examine results for transmission through a material with relative constants close to -1. For a passive material, the imaginary parts of ϵ_r and μ_r are negative (the engineer's convention). We treat the transmission of a delta-function line source through a layer. This source includes all spatial wave numbers. The longitudinal component k_z of the propagation vector normal to the surface assumes values that are negative real (corresponding to all angles from normal to grazing) and imaginary for the evanescent modes. Transmission in a medium of $\epsilon_r = \mu_r = -1$ amplifies the imaginary k_z terms and, thus, restores the evanescent waves and bypasses the usual diffraction limit of an ordinary lens. We show that small deviations from $\epsilon_r = \mu_r = -1$ cause a change from amplification to attenuation of these evanescent waves and thus limit the degree of improvement of an image.

Keywords: negative refractive index, perfect lens, double negative medium, transmission coefficient

1. INTRODUCTION

We have been attracted to work on problems involved in the design of substances with specified permittivity ϵ and permeability μ. These substances have been called, variously, metameterials or photonic crystals. Smith et al. made a metamaterial[1] with simultaneously negative ϵ and μ. J.B. Pendry[2] showed, theoretically, that a plane layer of material with relative permittivity ϵ_r and relative permeability μ_r both equal to -1 will transmit a perfect image of a flat object. Recently, Ziolkowski and Heyman[3] published a thorough investigation of wave propagation and focusing properties of a slab of a double negative medium (DNG) with emphasis on a time-domain point of view with a lossy Drude model. We present limitations on focusing at a single frequency by examining details of the transmission coefficient.

In 1968 V.G. Veselago[4] studied properties of DNG materials before any such material had been created. He made arguments for the possibility of such a material and pointed out unusual effects (e.g., reversed Doppler shift) of such a material. All of these effects follow from the result that the index of refraction is negative. Following Pendry's paper[2] there has been some discussion of how a DNG material leads to negative index of refraction. The result that $n = (\epsilon_r \mu_r)^{1/2}$ is negative for ϵ_r and μ_r negative follows from the requirement for a passive substance and consideration of the limit of a slightly lossy substance, as we shall show in more detail below. A layer of $n = -1$ material bends rays back so that the angle of refraction is the negative of the angle of reflection, but it also has the unusual feature that evanescent waves are focused with enhanced amplitude to provide a perfect image. The geometric condition for imaging is that the sum of the object distance from the incident side of the layer and the image distance from the other side be equal to the thickness of the layer.

We calculate in some detail the transmission of a wave from a 2-dimensional line source through a layer with ϵ_r and μ_r close to -1. For this 2-D problem, the electromagnetic propagation can be completely separated into a p-polarization (incident E field parallel to the plane of incidence) and s-polarization (incident E field perpendicular (senkrecht) to the plane of incidence). A delta function line source images to a delta function in the specified limit. However, for slight deviations from the limit, the magnitude of the transmission coefficient for the evanescent spectrum peaks and then starts to fall off exponentially (instead of continuing its exponential rise).

The 2-dimensional electromagnetic problem of reflection and transmission from a homogeneous uniform layer has been presented in many texts[5,6] . From this solution we can write the transmission coefficient for a plane wave in the following form, which is equivalent to equation (20) of Pendry.[2]

$$T = \frac{1}{\cos(k_{z2}d_2) + \frac{j}{2}\left(\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2}\right)\sin(k_{z2}d_2)} \tag{1}$$

where

$$\begin{aligned}
d_2 &= \text{thickness of layer of with permittivity } \epsilon_2 \text{and permeability } \mu_2 \\
k_{z2} &= (\epsilon_{2r}\mu_{2r}{k_o}^2 - {k_x}^2)^{1/2} = -j({k_x}^2 - {n_2}^2{k_o}^2)^{1/2} \\
Z_2 &= \text{wave impedance in medium 2 (the layer)} \\
Z_1 &= \text{wave impedance in medium 1, thefree space on either side of the layer.}
\end{aligned}$$

The plane of incidence is taken as the X-Z plane, with the material boundaries at $z = 0$ and $z = d_2$. The wave propagation has a positive Z component. The free-space wave number $k_o = \omega/c$, where ω is the angular frequency and c the speed of light. We have introduced the relative constants ϵ_{2r} and μ_{2r} and the index of refraction $n_2 = (\epsilon_{2r}\mu_{2r})^{1/2}$ in medium 2. We use the engineer's convention with time variation of cw components proportional to $e^{j\omega t}$. Thus, for a passive medium, ϵ_{2r} and μ_{2r} have negative imaginary parts and the branch of the square root for n_2 must also be chosen to give a negative imaginary part. The wave impedances have the following forms.

$$\begin{aligned}
Z_m &= \frac{k_{zm}}{\omega\epsilon_m}, &\quad \text{s-polarization.} \\
Z_m &= \frac{\omega\mu_m}{k_{zm}}, &\quad \text{p-polarization.}
\end{aligned}$$

with $m = 1$ in free space and $m = 2$ in the layer. In this general form, since the transmission coefficient T depends on the symmetric combination $Z_2/Z_1 + Z_1/Z_2$, the p-polarization and s-polarization cases will have similar forms for T, with the change of ϵ_{2r} to μ_{2r} in key places.

To be more explicit $\epsilon_1 = \epsilon_o$, $\mu_1 = \mu_o$, and $k_{z1} = ({k_o}^2 - {k_x}^2)^{1/2}$. Also, for p polarization

$$\frac{Z_2}{Z_1} = \frac{1}{\epsilon_{2r}}\frac{k_{z2}}{k_{z1}} = \frac{1}{\epsilon_{2r}}\frac{({n_2}^2{k_o}^2 - {k_x}^2)^{1/2}}{({k_o}^2 - {k_x}^2)^{1/2}},$$

and for s polarization

$$\frac{Z_1}{Z_2} = \frac{1}{\mu_{2r}}\frac{k_{z2}}{k_{z1}} = \frac{1}{\mu_{2r}}\frac{({n_2}^2{k_o}^2 - {k_x}^2)^{1/2}}{({k_o}^2 - {k_x}^2)^{1/2}}.$$

We treat the p-polarization case explicitly here. The selection criterion for the correct branch of these square roots is to choose signs that give attenuation of any wave (real or evanescent) in the two media.

2. REFRACTIVE INDEX CLOSE TO -1

2.1. To show DNG medium has negative refractive index

We are considering cases of n_2 close to -1. More precisely, $\epsilon_{2r} = -\epsilon'_{2r} - j\epsilon''_{2r}$ and $\mu_{2r} = -\mu'_{2r} - j\mu''_{2r}$, where both ϵ'_{2r} and μ'_{2r} are real and close to 1, and where both ϵ''_{2r} and μ''_{2r} are real, positive, and small compared to 1. Thus,

$${n_2}^2 = \epsilon'_{2r}\mu'_{2r} - \epsilon''_{2r}\mu''_{2r} + j\left(\epsilon'_{2r}\mu''_{2r} + \mu'_{2r}\epsilon''_{2r}\right) = a + j\delta_2, \tag{2}$$

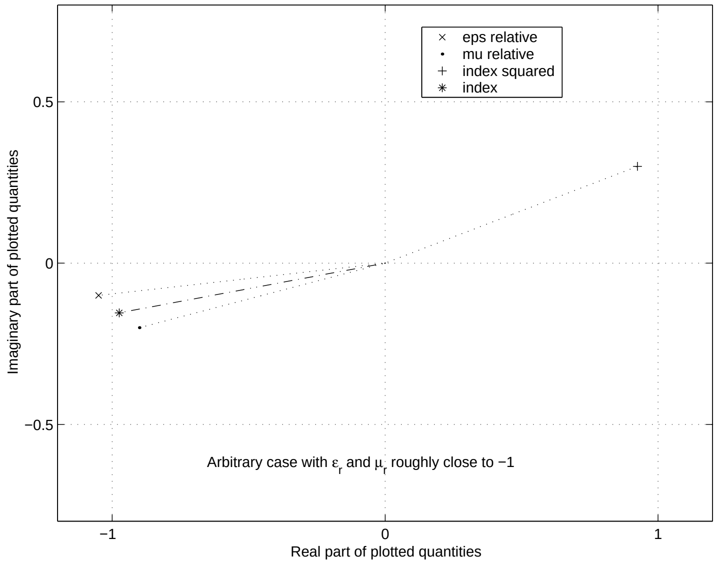

Figure 1. Illustration of permittivity, permeability and index of refraction for a slightly lossy material with dominant negative real parts of permittivity and permeability.

where (2) serves to define a and δ_2. By the choice of cases a is real and close to 1, and δ_2 is real, positive, and small of the same order as ϵ''_{2r} and μ''_{2r}. The index of refraction n_2 is a square root of (2). With the assumed range of values for a and δ_2, then n_2 lies either in the first quadrant (same as ${n_2}^2$) or of opposite sign in the third quadrant. We know that a passive medium requires a negative imaginary part of n_2; hence, the choice must be the third quadrant branch where the real part is negative. This logic is our proof that n_2 is negative in a DNG medium. Figure 1 illustrates these quantities in the complex plane.

2.2. Behavior of k_{z1} and k_{z2}

Now we show the choice of branch for k_{z2} for two cases: (a)$k_x < \sqrt{a}k_o$ and (b)$k_x > \sqrt{a}k_o$. The convention for the square root sign is the root that has a positive real part (ambiguous for real part = 0, but determined by continuation on the branch).

For case (a)

$$k_{z2} = \pm\sqrt{a{k_o}^2 - {k_x}^2 - i\delta_2{k_o}^2} \approx \pm\sqrt{a{k_o}^2 - {k_x}^2}\left(1 + \frac{j}{2}\frac{\delta_2{k_o}^2}{(a{k_o}^2 - {k_x}^2)}\right).$$

The approximation becomes poor as $k_x \to \sqrt{a}k_o$, but the square root stays in the same quadrant. Since k_{z2} must have a negative imaginary part,

$$k_{z2} = -\sqrt{{n_2}^2{k_o}^2 - {k_x}^2}\,. \tag{3}$$

For case (b)

$$k_{z2} = \pm j\sqrt{{k_x}^2 - a{k_o}^2 + i\delta_2{k_o}^2} \approx \pm j\sqrt{{k_x}^2 - a{k_o}^2}\left(1 + \frac{j}{2}\frac{\delta_2{k_o}^2}{({k_x}^2 - a{k_o}^2)}\right).$$

Again, with this separation into a real and imaginary part with known signs, the choice of the overall minus sign keeps the imaginary part negative, and

$$k_{z2} = -j\sqrt{{k_x}^2 - {n_2}^2{k_o}^2}\,. \tag{4}$$

With the same type of analysis, the longitudinal wave number in free space follows the branches given by:
(a) For $k_x < k_o$

$$k_{z1} = \sqrt{{k_o}^2 - {k_x}^2} \to k_o \text{ as } k_x \to 0\,.$$

(b) For $k_x > k_o$

$$k_{z1} = -j\sqrt{{k_x}^2 - {k_o}^2} \to -jk_x \text{ as } k_x \to \infty\,.$$

The derivation follows by considering the limit of a medium with arbitrarily small loss. We summarize the result thus: in the propagating region, the k_z's are of opposite sign in the negative refracting medium relative to the normal refracting medium, whereas in the evanescent region the signs are the same. In the complex plane, k_{z2} lies in the third quadrant and k_{z1} lies in the fourth quadrant. Figure 2 shows a plot of their paths. Both paths begin infinitely far down in the complex plane as k_x increases from $-\infty$, track to the points indicated with a "$*$" for k_{z1} and an "x" for k_{z2} at $k_x = 0$, and then retrace the incoming paths as $k_x \to \infty$.

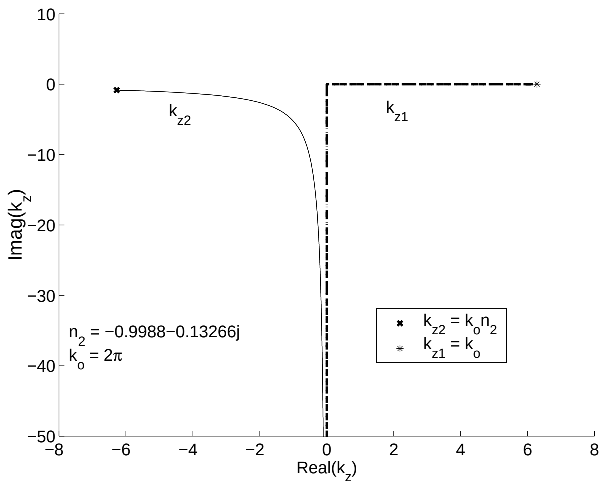

Figure 2. Complex plane paths of longitudinal wavenumbers k_{z1} and k_{z2} as a function of k_x for a slightly lossy DNG medium.

3. TRANSMISSION COEFFICIENT

From these considerations, it becomes clear that
(a) For k_x fairly small compared to k_o

$$\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2} = 2 + R_s\,.$$

(b) For k_x fairly large compared to k_o

$$\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2} = -2 + R_b\,.$$

R_s and R_b have a small magnitude of second order in the deviations of ϵ_{2r} and μ_{2r} from -1. The meaning of fairly large depends on the size of these deviations, but the region of validity of small R_s and small R_b grows to include values closer to k_o as the deviations approach zero.

The use of this expression for $\frac{Z_2}{Z_1} + \frac{Z_1}{Z_2}$ in (1) leads to the forms:
(a) For $k_x < \sqrt{a}k_o$ and $k_x < k_o$ (Recall $a \approx 1$.)

$$T = \frac{1}{e^{+jk_{z2}d_2} + \frac{R_s}{4}\left(e^{+jk_{z2}d_2} - e^{-jk_{z2}d_2}\right)} \approx e^{-jk_{z2}d_2}\left[1 - \frac{R_s}{4}\left(1 - e^{-2jk_{z2}d_2}\right)\right] \tag{5a}$$

$$\rightarrow e^{+jk_{z1}d_2} \qquad \text{as } n_2 \rightarrow -1 \tag{5b}$$

because $k_{z2} \rightarrow -k_{z1}$ in this region.
(b) For $k_x > \sqrt{a}k_o$ and $k_x > k_o$

$$T = \frac{1}{e^{-jk_{z2}d_2} + \frac{R_b}{4}\left(e^{+jk_{z2}d_2} - e^{-jk_{z2}d_2}\right)} \approx e^{jk_{z2}d_2}\left[1 - \frac{R_b}{4}\left(e^{2jk_{z2}d_2} - 1\right)\right] \tag{6a}$$

$$\rightarrow e^{jk_{z1}d_2} \qquad \text{as } n_2 \rightarrow -1\,. \tag{6b}$$

In case (b), the region of evanescent waves where the k_z's have a predominant imaginary part, (6a) becomes

$$T = \frac{\exp\left(d_2\sqrt{k_x{}^2 - k_o{}^2}\right)}{1 + \frac{R_b}{4}\left[\exp\left(2d_2\sqrt{k_x{}^2 - k_o{}^2}\right) - 1\right]} \approx \frac{exp\left(d_2\sqrt{k_x{}^2 - k_o{}^2}\right)}{1 + \frac{R_b}{4}\left[\exp\left(2d_2\sqrt{k_x{}^2 - k_o{}^2}\right)\right]}\,.$$

For our cases where R_b is quite small, $|T|$ has the property of growing in proportion to $\exp\left(d_2\sqrt{k_x{}^2 - k_o{}^2}\right)$ until this exponential gets to be about as large as $\sqrt{R_b}/2$. $|T|$ peaks in this vicinity. Then, as this exponential gets large compared to $\sqrt{R_b}/2$,

$$|T| \sim \left(4/|R_b|\right)\exp\left(-d_2\sqrt{k_x{}^2 - k_o{}^2}\right). \tag{7}$$

where this approximation becomes an asymptote as $k_x \rightarrow \infty$.

We have programmed complete equations for the transmission coefficient and show the behavior of $|T|$ as a function of k_x/k_o for four sequences of ϵ_{2r} and μ_{2r} in Figure 3.

Furthermore, it is interesting that to lowest order (which is second order) in the deviations of ϵ_{2r} and μ_{2r} from -1 both R_s and R_b have the same form.

$$R_s \approx R_b \approx \left(\delta_e + \frac{\delta_c}{2}\frac{k_o{}^2}{k_o{}^2 - k_x{}^2}\right)^2, \tag{8}$$

where $\delta_e = \epsilon_{2r} + 1$ and $\delta_c = \epsilon_{2r}\mu_{2r} - 1$.

All the graphs in Figure 3 include the curve for the perfect focusing case, $\epsilon_{2r} = \mu_{2r} = -1$. The graph 3(a) also shows the curve that is the result of no layer at all; i.e., where $\epsilon_{2r} = \mu_{2r} = +1$. The magnitude of the propagating part of the transmitted wave is the same for both of these cases, but with no layer the evanescent spectrum shows its "standard" (for a point source) exponentially decreasing amplitude factor in propagation.

We also calculated phase curves for all of these cases. The phase is difficult to follow in these cases, but the convergence to the $\epsilon_{2r} = \mu_{2r} = -1$ case does occur similarly out to the turn-over value of k_x.

4. SPECTRUM OF A LINE SOURCE AND ITS IMAGE

We consider a line magnetic source at a position $z = -d_1$ relative to the layer of negative refractive material between the planes $z = 0$ and $z = d_2$. The geometry and coordinates are illustrated (Layer is truncated in X direction.) in Figure 4. The following paragraphs show the δ-function spectrum of the E-field of the source and derive the spectrum of the E-field in the image plane. For the perfect lens material of $\epsilon_{2r} = \mu_{2r} = -1$, these spectra are identical.

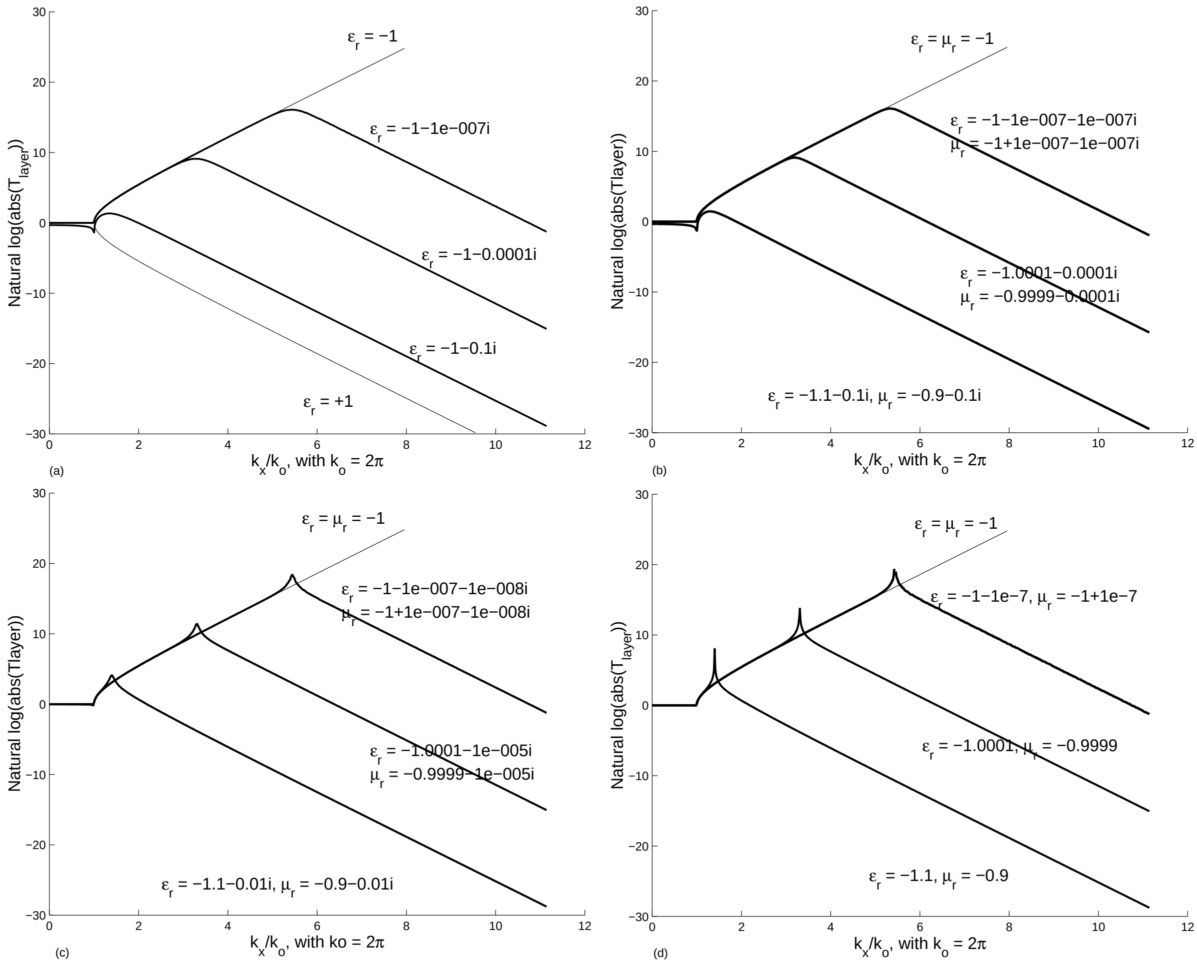

Figure 3. Transmission coefficient for propagating and evanescent parts of the spectra through a half-wavelength layer, p polarization. In 3(a), each case has $\mu_r = \epsilon_r$ and deviations from -1 are only in the imaginary part. In 3(b), the deviation in ϵ_r is of the form $\delta(-1-j)$, and the deviation in mu_r is of the form $\delta(1-j)$. In 3(c) the deviation in ϵ_r is of the form $\delta(-1-0.1j)$, and the deviation in mu_r is of the form $\delta(1-0.1j)$. In 3(d) the deviations are only in the real part, and of the form $-\delta$ for ϵ_r and δ for mu_r. The small parameter δ is positive.

A magnetic current source of $I_m\delta(x)\delta(z+d_1)$ in the Y direction (out of the paper in Figure 4) is chosen because it drives a p-polarized wave. The source magnetic field satisfies the 2-D wave equation outside this line source.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+{k_o}^2\right)H_y = j\omega\epsilon_o J_m = -j\omega\epsilon_o I_m\big(-\delta(x)\delta(z+d_1)\big).$$

The source magnetic field is, thus, a scalar multiple $-j\omega\epsilon_o I_m$ of the Green's function.

$$H_y(x,z) = -j\omega\epsilon_o I_m\left(\frac{-j}{4}{H_o}^{(2)}(k_o|\bar{\rho}+d_1\hat{z}|)\right)$$

where $\bar{\rho} = x\hat{x}+z\hat{z}$, $\hat{x}$ and $\hat{z}$ are unit coordinate vectors, and ${H_o}^{(2)}$ is the zero-order Hankel function of the second kind. This solution can also be expressed in a Fourier transform as

$$H_y(x,z) = -\frac{\omega\epsilon_o I_m}{4\pi}\int_{-\infty}^{\infty}\frac{\exp\left(-jk_x x - jk_{z1}|z+d_1|\right)}{k_{z1}}\mathrm{d}k_x.$$

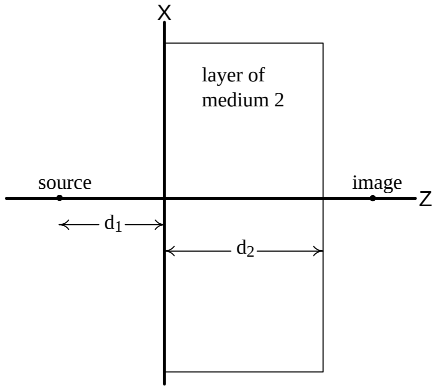

Figure 4. Coordinate system and location of the source and image relative to the DNG layer.

The electric field of this wave is obtained from Ampere's law, and in particular the transverse incident E field in the region $z > -d_1$ is

$$E_x^{inc} = -\frac{1}{j\omega\epsilon_o}\frac{\partial H_y}{\partial z} = -\frac{I_m}{4\pi}\int_{-\infty}^{\infty} \exp\big(-jk_x x - jk_{z1}(z+d_1)\big)\mathrm{d}k_x.$$

In the plane $z = -d_1$, this gives $E_x^{inc} = \big(-I_m/2\big)\delta(x)$. Furthermore, in the region $z > d_2$, the fields have only a transmitted component with

$$E_x^{tr} = \frac{-I_m}{4\pi}\int_{-\infty}^{\infty} T(k_x)\exp\big(-jk_x x - jk_{z1}(z-d_2+d_1)\big)\mathrm{d}k_x$$

where $T(k_x)$ is given by (1), and more explicitly by (5a) and (6a). In the limit that $\epsilon_{2r} = \mu_{2r} = -1$, $T(k_x) = exp(jk_{z1}d_2)$ (where ${k_x}^2 + k_{z1}^2 = {k_o}^2$), and thus

$$E_x^{tr} = \frac{-I_m}{4\pi}\int_{-\infty}^{\infty}\exp\big(-jk_{z1}(z-2d_2+d_1) - jk_x x\big)\mathrm{d}k_x.$$

As a restatement of Pendry's conclusion,[2] in the plane $z = 2d_2 - d_1$ (and in the region where $z > d_2$, so for a focal plane to exist, $d_2 > d_1$), we have a delta-function image of the delta-function source and so by superposition a perfect image of any flat source.

We examine the lack of perfection for the cases where ϵ_{2r} and μ_{2r} deviate slightly from -1 by examining in further detail the spectrum of E_x^{tr}. If we let $I_m = -2$ and consider the factor $1/(2\pi)$ to be associated with the inverse transform definition, then the spectrum in the region $z > d_2$ is

$$\widetilde{E}(k_x) = T(k_x)\exp\big(-jk_{z1}(z-d_2+d_1)\big)\,.$$

We generated calculations of $\widetilde{E}(k_x)$ in the image plane. Plots of the results are shown in Figure 5. A perfectly focusing layer would produce a flat line of magnitude 1 (logarithm = 0) in the graphs of Figure 5. We notice that the focusing loses accuracy for the evanescent waves at about the same values of k_x for comparable absolute values of the deviations from -1. When the deviations are real, or quite close, there is a spike in the amplitude just before it starts to decrease.

For large k_x, $\ln\left|T\exp(-jk_{z1}d_2)\right|$ is asymptotic to a straight line with slope $-2d_2$. From (7) and (8) the asymptotic form is

$$\left|T\big(\exp(-jk_{z1}d_2)\big)\right| \;\sim\; \frac{4}{|\delta_e|^2}\exp(-2k_x d_2)\,.$$

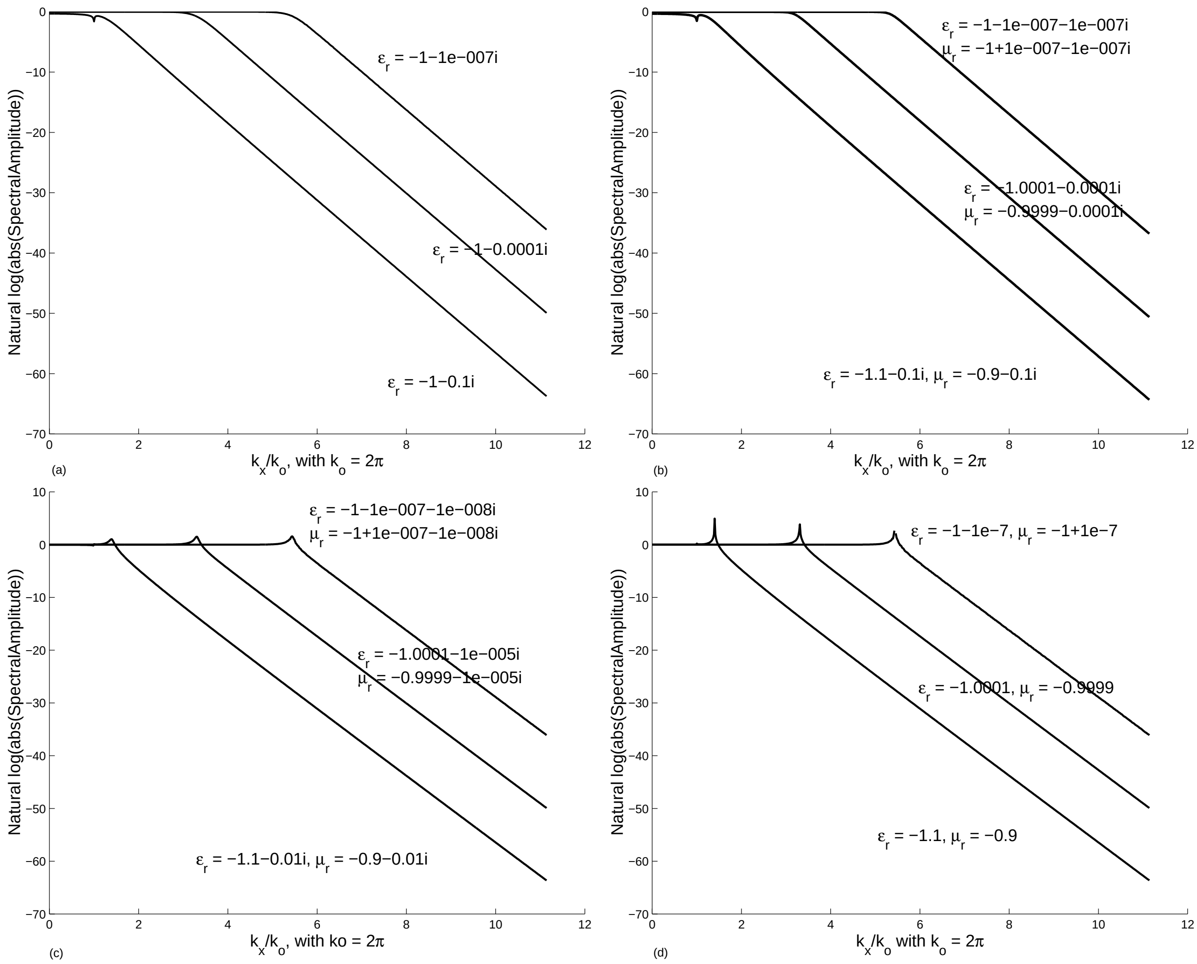

Figure 5. Log of magnitude of spectrum of line source image through a half-wavelength layer, p polarization. In 5(a), each case has $\mu_r = \epsilon_r$ and deviations from -1 are only in the imaginary part. In 5(b), the deviation in ϵ_r is of the form $\delta(-1-j)$, and the deviation in mu_r is of the form $\delta(1-j)$. In 5(c) the deviation in ϵ_r is of the form $\delta(-1-0.1j)$, and the deviation in mu_r is of the form $\delta(1-0.1j)$. In 5(d) the deviations are only in the real part, and of the form $-\delta$ for ϵ_r and δ for mu_r. The small parameter δ is positive.

As the plots of Figure 5 illustrate, $\ln\left|Te^{-jk_{z1}d_2}\right| \to 0$ for $k_x < k_o$. The shoulder (half-width) of this spectrum may then be regarded as the point where $|T| = 1$ as given by (7) with the approximation from (8) that $R_b = {\delta_e}^2$. Let Δk denote the full width and $M_1 = \ln\left(2/|\delta_e|\right)$. Then

$$\Delta k/2 = \sqrt{(M_1/d_2)^2 + {k_o}^2}\ .$$

The focal spot width ΔW (full width at half height) is then related inversely to Δk roughly by $\Delta W \Delta k \approx 2\pi$. Calculations based on the asymptotic spectrum verify this formula is accurate within 20% for $d_2 = 1$.

5. CONCLUSIONS

Materials designed to provide negative index of refraction will have unusual and surprising reflection, refraction, and transmission properties. One of these is the possibility to create an image with a planar layer. But this image will have many practical limits, which will be related to the fact that such materials will have a loss component, will have to be carefully designed to get ϵ_{2r} and μ_{2r} close to -1, and will have strong frequency dispersion. From the above, we note that the perfect lens occurs only when ϵ_{2r} and μ_{2r} are exactly equal to -1, and the image is so sensitive that a slight deviation of ϵ_{2r} and μ_{2r} from -1 causes a significant departure from the perfect image.

In conclusion we agree with Pendry's calculation[2] of the limit of the transmission coefficient, which yields amplification of the evanescent waves to provide a perfect image. We examined the limiting behavior of a passive medium as it approaches this limit of $\epsilon_{2r} = \mu_{2r} = -1$. For a given ϵ_{2r} and μ_{2r} (both very close to -1), then for $|k_x|$ less than some "turn-around" value the transmission coefficient is close to the ideal amplifying factor. However, for larger $|k_x|$, T rapidly diverges from that ideal perfect-focus amplification.

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REFERENCES

1. D. R. Smith, Willie J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, "Composite Medium with Simultaneously Negative Permeability and Permittivity," *Phys. Rev. Lett.*, **84**, pp. 4184–4187, 2000.
2. J. B. Pendry, "Negative Refraction Makes a Perfect Lens," *Phys Rev. Lett.*, **85**, pp. 3966–3969, 2000.
3. Richard W. Ziolkowski and Ehud Heyman, "Wave propagation in media having negative permittivity and permeability," *Phys. Rev. E*, **64**, pp. 056625-1–056625-15, 2001.
4. V. G. Veselago, "The Electrodynamics of Substances with Simultaneously Negative Values of ϵ and μ," *Sov. Phys. Usp.* **10**, pp. 509-514, 1968.
5. Akira Ishimaru, *Electromagnetic Wave Propagation, Radiation and Scattering*, pp. 36–45, Prentice Hall, New Jersey, 1991.
6. Weng-Cho Chew, *Waves and Fields in Inhomogeneous Media*, pp. 48-53, IEEE Press, New York, 1995.