Lucent Technologies

How to Generate Passive Reduced-Order Models

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Outline

- Traditional circuit simulation
- Interconnect dominates
- Need for reduced-order modeling
- “Optimal” reduced-order models via a Lanczos-type algorithm
- Passivity
- Passive reduced-order models via projection
- Open problems
- Concluding remarks

Traditional lumped-circuit model

- Only the circuit elements (resistors, capacitors, inductors, transistors,...) are taken into account
- The wires that connect the circuit elements, the so-called *interconnect*, could be ignored or replaced by very simple models
- Three types of equations:
  Kirchhoff's current law, Kirchhoff's voltage law, constitutive relations (e.g., Ohm's law for a resistor)
- All these equations can be summarized as a system of first-order differential algebraic equations (DAEs):

$$\frac{d}{dt}\mathbf{q}(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) = \mathbf{0}$$

- Length of the vector $\mathbf{x}$ is of the order of the number of circuit elements

Interconnect can no longer be ignored

- Size of circuit elements keeps decreasing
- Signal speeds keep increasing
- The interconnect dominates the timing behavior of the circuit
- Each (small) piece of interconnect is modeled as a linear time-invariant RCL subcircuit:

$$\mathbf{C}\,\frac{d\,\mathbf{x}}{d\,t} + \mathbf{G}\,\mathbf{x} = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\top}\mathbf{x}(t)$$

- Each such subcircuit can have up to $\mathcal{O}(10^6)$ elements
- There can be up to $\mathcal{O}(10^6)$ such subcircuits
- RCL subcircuits are *passive*

Linear dynamical systems

- DAE of the type

$$\mathbf{C}\,\frac{d\,\mathbf{x}}{d\,t} + \mathbf{G}\,\mathbf{x} = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{E}^{\top}\,\mathbf{x}(t)$$

  where $\mathbf{C}$, $\mathbf{G} \in \mathbb{C}^{N \times N}$, $\mathbf{B} \in \mathbb{C}^{N \times p}$, $\mathbf{E} \in \mathbb{C}^{N \times m}$

- p inputs, m outputs
- $\mathbf{G} + s\,\mathbf{C}$ is a regular pencil
- N is large
- For RCL subcircuits: $\mathbf{E} = \mathbf{B}$, $m = p$
- From now on, assume that $\mathbf{E} = \mathbf{B}$ and $m = p$

Reduced-order modeling

- Replace each subcircuit by *reduced-order model*:

$$\mathbf{C}_n \frac{d\,\mathbf{z}}{d\,t} + \mathbf{G}_n\,\mathbf{z} = \mathbf{B}_n\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}_n^\top \mathbf{z}(t)$$

- $\mathbf{G}_n$ and $\mathbf{C}_n$ are $n \times n$ matrices and $\mathbf{B}_n$ is an $n \times p$ matrix, where $n \ll N$ ($n \approx 10^{0-2}$)
- Typically, n is a small multiple of p
- $\mathbf{G}_n$, $\mathbf{C}_n$, and $\mathbf{B}_n$ are chosen such that

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^q\right)$$

where $\mathbf{H}(s) = \mathbf{B}^\top\left(\mathbf{G} + s\,\mathbf{C}\right)^{-1}\mathbf{B}$ and $\mathbf{H}_n(s) = \mathbf{B}_n^\top\left(\mathbf{G}_n + s\,\mathbf{C}_n\right)^{-1}\mathbf{B}_n$ are the frequency-domain *transfer functions* of the subcircuit and its reduced-order model

Matrix-Padé reduced-order models

- Optimal models: $q = q(n)$ is as large as possible
- More general case: $\mathbf{H}(s) = \mathbf{L}^{\mathsf{T}} (\mathbf{G} + s\,\mathbf{C})^{-1}\,\mathbf{B}$
  where $\mathbf{G}$, $\mathbf{C}$ are $N \times N$, $\mathbf{B}$ is $N \times p$, and $\mathbf{L}$ is $N \times m$
- **MPVL** (Feldmann and F.,'95):
  - Rewrite: $\mathbf{H}(s) = \mathbf{L}^{\mathsf{T}} (\mathbf{I} + (s - s_0)\,\mathbf{A})^{-1}\,\mathbf{R}$
    where $\mathbf{A} = (\mathbf{G} + s_0\,\mathbf{C})^{-1}\,\mathbf{C}$ and $\mathbf{R} = (\mathbf{G} + s_0\,\mathbf{C})^{-1}\,\mathbf{B}$
  - Run n steps of Lanczos-type algorithm applied to $\mathbf{A}$ with and right and left starting vectors $\mathbf{R}$ and $\mathbf{L}$
  - Matrix-Padé reduced-order model:

$$\mathbf{H}_n(s) = \mathbf{L}_n^{\mathsf{T}} (\mathbf{I} + (s - s_0)\,\mathbf{T}_n)^{-1}\,\mathbf{R}_n$$

- $q(n) \geq \lfloor n/p \rfloor + \lfloor n/m \rfloor$

Back to RCL subcircuits

- Recall: $\mathbf{H}(s) = \mathbf{B}^\top (\mathbf{G} + s\,\mathbf{C})^{-1}\mathbf{B}$
- Additional structure:

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{12}^\top & \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

  where $\mathbf{G}_{11}, \mathbf{C}_{11}, \mathbf{C}_{22}$ are symmetric positive semidefinite

- "Symmetric" algorithm: **SyMPVL**
  (Feldmann and F.,'97 and '98)
- **SyMPVL** based on coupled recurrences
  (Bai and F., '01)

SyMPVL for RCL subcircuits

- Recall: $\mathbf{H}(s) = \mathbf{B}^\top (\mathbf{G} + s\,\mathbf{C})^{-1}\,\mathbf{B}$
- Select real expansion point s_0
- Reduce to "essentially" one matrix
- Compute factorization

$$\mathbf{G} + s_0\,\mathbf{C} = \mathbf{U}^\top \mathbf{J}\,\mathbf{U}$$

where $\mathbf{J} = \mathbf{J}^\top$ is "simple" (for RC, RL, LC circuits: $\mathbf{J} = \mathbf{I}$)

- With this factorization:

$$\begin{aligned}\mathbf{H}(s) &= \mathbf{B}^\top \left(\mathbf{U}^\top \mathbf{J}\,\mathbf{U} + (s - s_0)\,\mathbf{C}\right)^{-1}\mathbf{B}\\ &= \mathbf{R}^\top (\mathbf{J} + (s - s_0)\,\mathbf{A})^{-1}\,\mathbf{R}\end{aligned}$$

where $\quad \mathbf{A} = \mathbf{U}^{-\top}\,\mathbf{C}\,\mathbf{U}^{-1} = \mathbf{A}^\top \quad$ and $\quad \mathbf{R} = \mathbf{U}^{-\top}\,\mathbf{B}$

SyMPVL for RCL subcircuits, continued

- Transfer function of RCL subcircuit:

$$\mathbf{H}(s) = \mathbf{R}^{\top} \left(\mathbf{J} + (s - s_0)\, \mathbf{A}\right)^{-1} \mathbf{R}$$

- Apply $\mathbf{J}$-symmetric Lanczos-type method to the symmetric matrix $\mathbf{A}$ and the block $\mathbf{R}$ of p starting vectors

- After n steps, the algorithm has generated n Lanczos vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

  that

  - are $\mathbf{J}$-orthogonal: $\mathbf{v}_j^{\top} \mathbf{J}\, \mathbf{v}_k = 0$ for all $j \neq k$

  - build a basis for the space spanned by the n first linearly columns of the block Krylov matrix

$$\left[\mathbf{J}^{-1}\mathbf{R} \;\; \left(\mathbf{J}^{-1}\mathbf{A}\right)\mathbf{J}^{-1}\mathbf{R} \;\; \cdots \;\; \left(\mathbf{J}^{-1}\mathbf{A}\right)^{i}\mathbf{J}^{-1}\mathbf{R} \;\; \cdots\right]$$

Lanczos in matrix form

- Matrix of first n Lanczos vectors:

$$\mathbf{V}_n = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

- $\mathbf{J}$-orthogonality:

$$\mathbf{V}_n^\top \mathbf{J} \mathbf{V}_n = \boxed{\mathbf{\Delta}_n} = \text{diagonal matrix}$$

- Corresponding projection of $\mathbf{A}$ onto Lanczos vectors:

$$\boxed{\mathbf{T}_n} = \boxed{\mathbf{\Delta}_n^{-1}} \boxed{\mathbf{V}_n^\top} \boxed{\mathbf{A}} \boxed{\mathbf{V}_n}$$

- $\mathbf{T}_n$ and $\mathbf{\Delta}_n$ are computed on the "fly"

SyMPVL for RCL subcircuits, continued

- During first p Lanczos steps, we $\mathbf{J}$-orthogonalize the starting vectors:

$$\mathbf{J}^{-1}\mathbf{R} = \mathbf{V}_p\,\boldsymbol{\rho}$$

- In terms of matrices $\mathbf{T}_n$, $\boldsymbol{\Delta}_n$, $\boldsymbol{\rho}$ from n Lanczos steps, the n-th Padé approximant $\mathbf{H}_n$ to $\mathbf{H}$ is given by

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\mathsf{T}}\left(\boldsymbol{\Delta}_n^{-1} + (s-s_0)\,\mathbf{T}_n\,\boldsymbol{\Delta}_n^{-1}\right)^{-1}\mathbf{B}_n, \quad \mathbf{B}_n = \begin{bmatrix} \boldsymbol{\rho} \\ \mathbf{0}_{n-p\times p} \end{bmatrix}$$

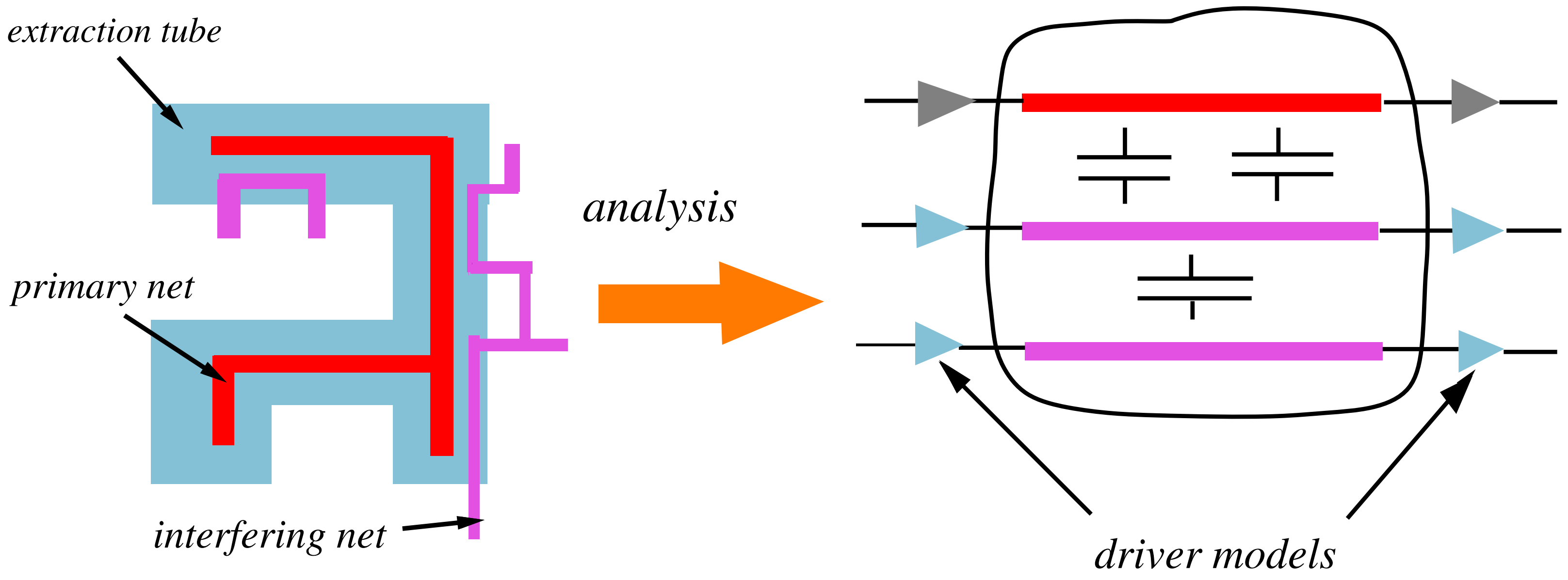
- For RC, RL, and LC circuits:

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\mathsf{T}}\left(\mathbf{I} + (s-s_0)\,\mathbf{T}_n\right)^{-1}\mathbf{B}_n$$

  and $\mathbf{T}_n$ is symmetric positive semidefinite

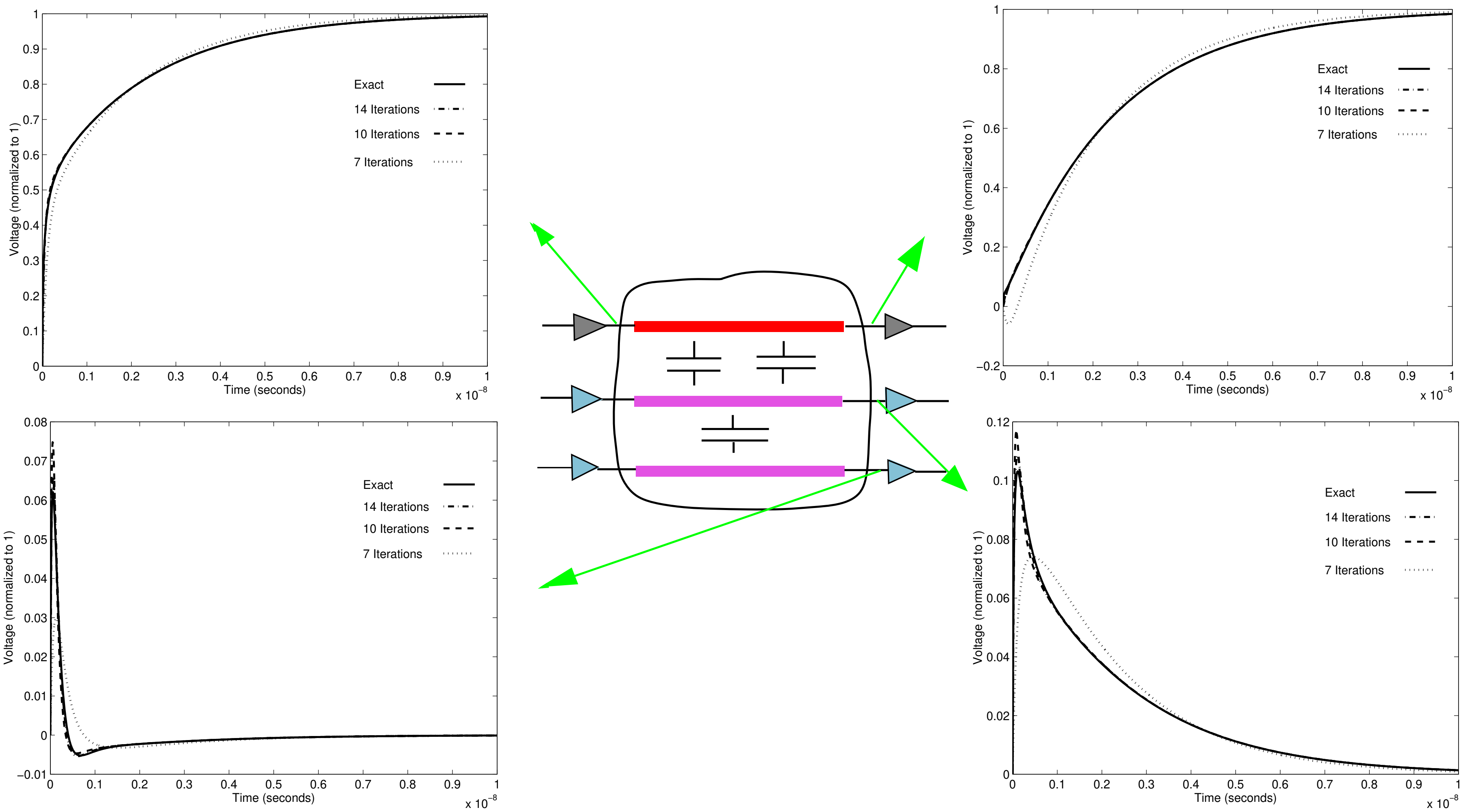
- $\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s-s_0)^q\right)$ where $q(n) \geq 2\lfloor n/p \rfloor$

Signal-integrity verification



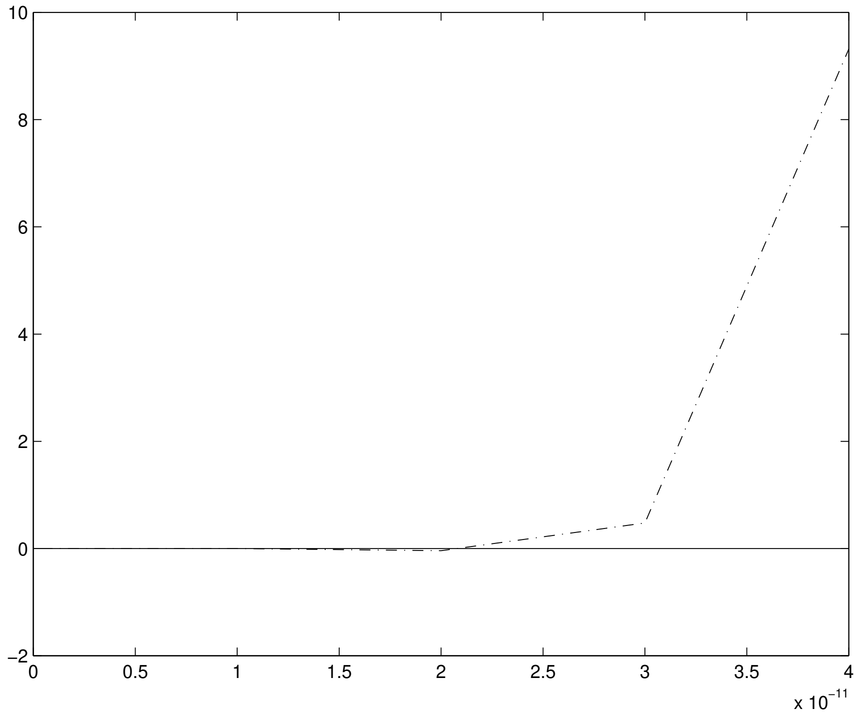
- Select clusters of potentially interfering nets
- Analyze large RC(LM) circuits plus drivers

Delay and cross-talk



Passivity

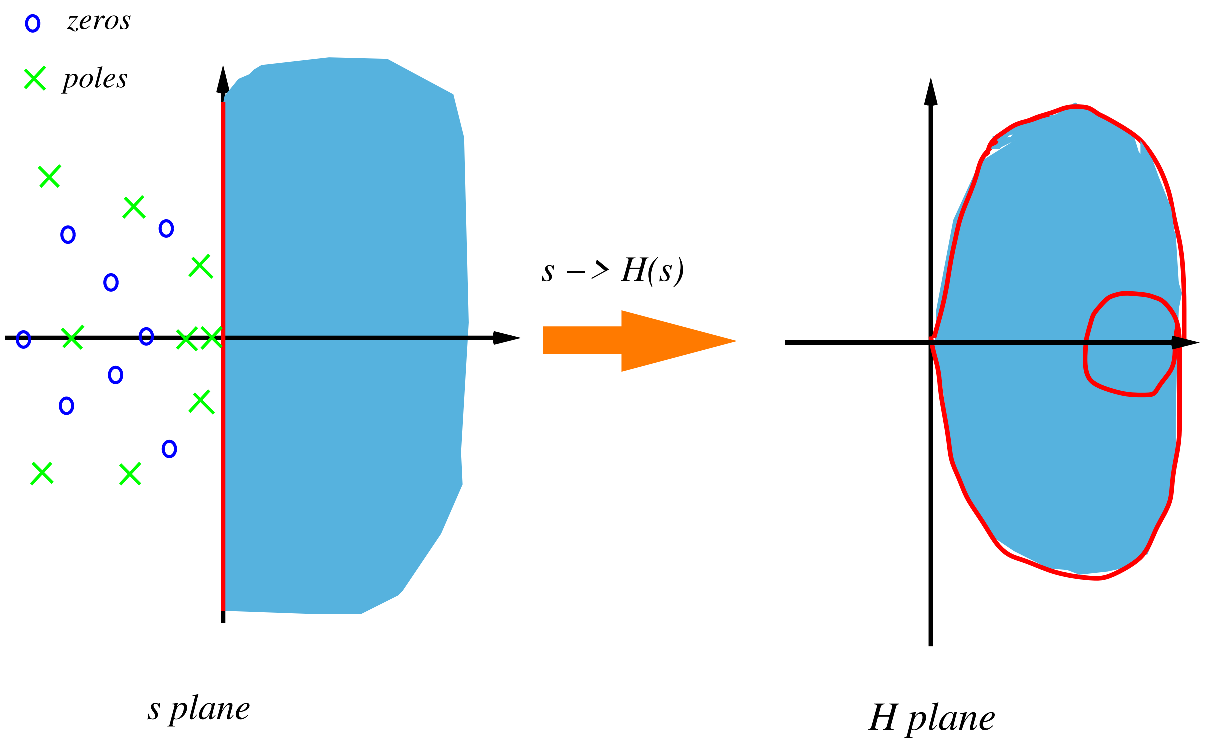
- A system is *passive* if it does not generate energy
- RCL subcircuits are passive
- Reduced-order model should preserve passivity
- Combining non-passive reduced-order models may result in *unstable* circuits



Passivity and positive realness

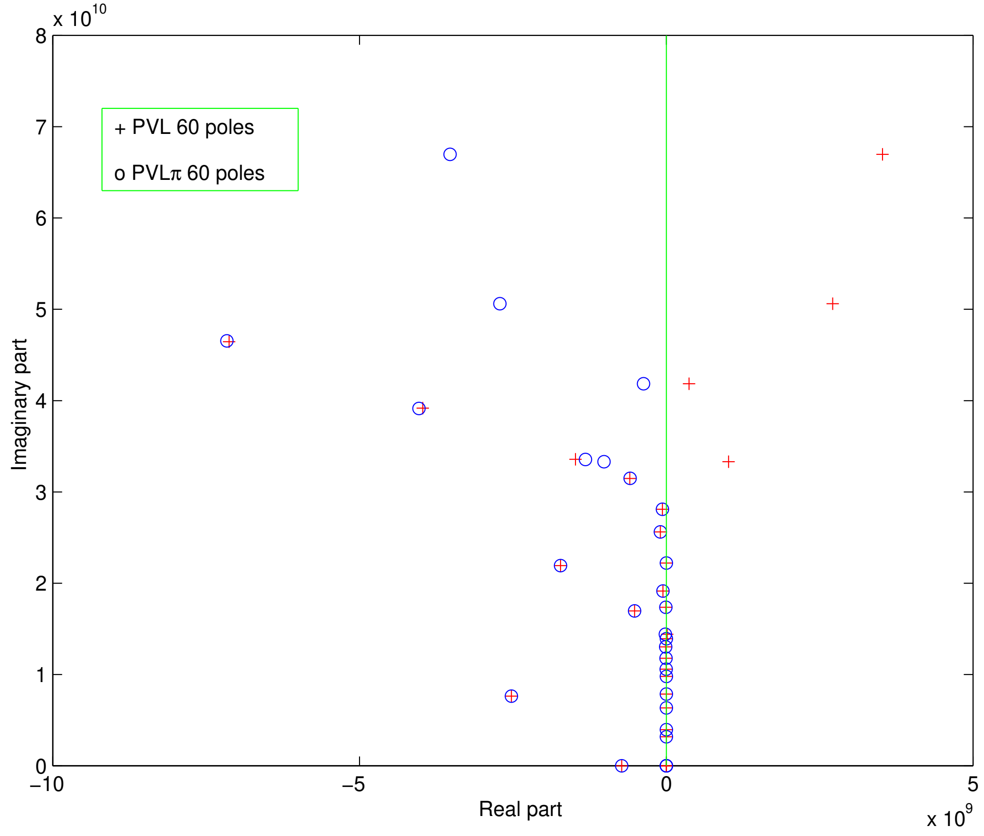
- A time-invariant linear dynamical system with p inputs and outputs is *passive* if, and only if, its transfer function $\mathbf{H}$ is *positive real*:
  - $\mathbf{H}$ has no poles in $\mathbb{C}_+ = \{\, s \in \mathbb{C} \mid \mathrm{Re}\, s > 0 \,\}$
  - $\mathbf{H}$ is real for real s
  - $\mathbf{H}(s) + (\mathbf{H}(s))^{\mathsf{H}} \geq \mathbf{0}$ for all $s \in \mathbb{C}_+$
- Passivity implies stability

Passivity (for $p = 1$)



$\left(2\,\mathrm{Re}\,\mathbf{H}(s) =\right) \mathbf{H}(s) + (\mathbf{H}(s))^{\mathsf{H}} \geq 0$ for all $s \in \mathbb{C}$ with $\mathrm{Re}\,s > 0$

Padé models do not preserve passivity



Poles of a Padé model and of a passive model

Passivity for special cases

- For RC, RL, and LC circuits:

$$\mathbf{H}_n(s) = \mathbf{B}_n^{\top} \left(\mathbf{I} + s\,\mathbf{T}_n\right)^{-1} \mathbf{B}_n$$

  and $\mathbf{T}_n$ is symmetric positive semidefinite
  (For simplicity: $s_0 = 0$)

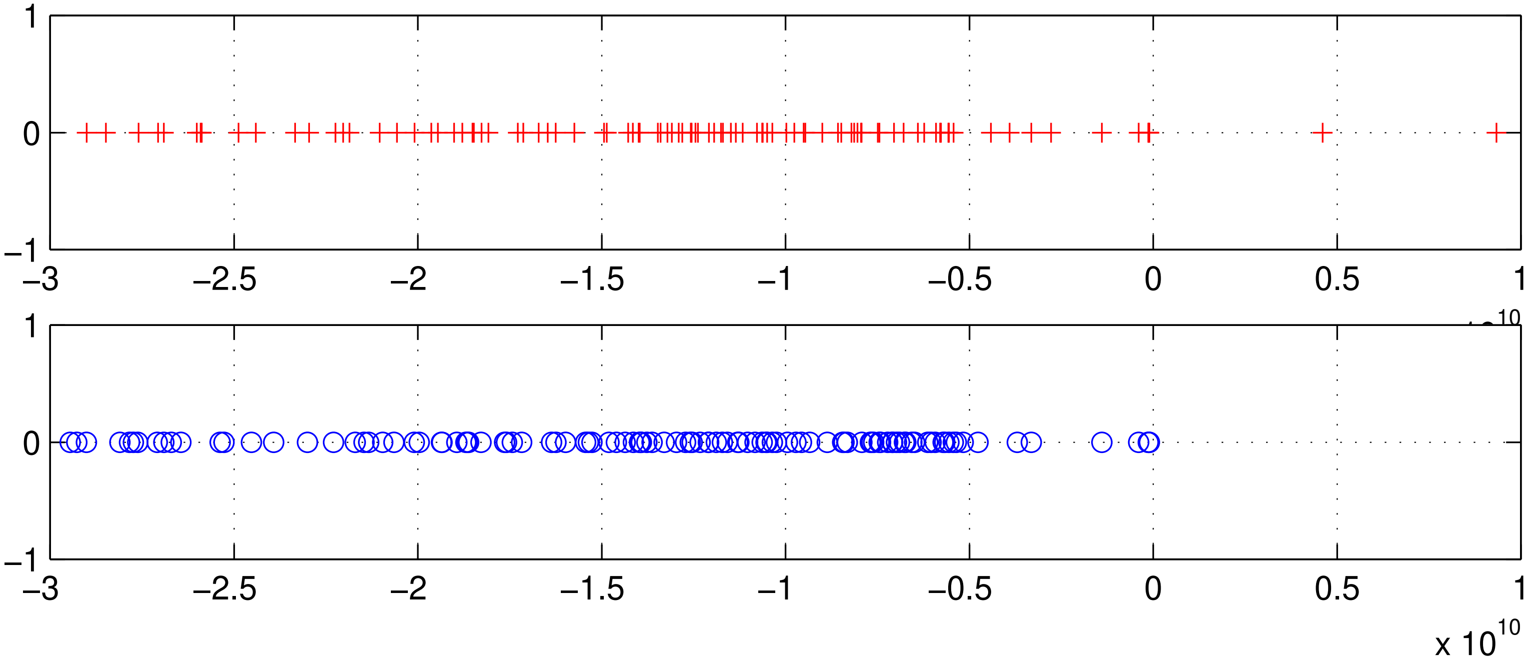
- Using the positive definiteness of $\mathbf{T}_n$, it is straightforward to verify that $\mathbf{H}_n$ is positive real and thus passive

- Need to guarantee that $\mathbf{T}_n$ is positive definite in finite-precision arithmetic

- The key is to implement the Lanczos method with coupled recurrences:
  compute Cholesky factor $\mathbf{L}_n$ and set $\mathbf{T}_n = \mathbf{L}_n \mathbf{L}_n^{\top}$

SyMPVL simulation of large RC circuit

- $\approx 200,000$ RC elements, $p = 150$ I/O ports
- Reduced-order model of order $n = 300$



Poles from non-coupled (+) and coupled (o) recurrences

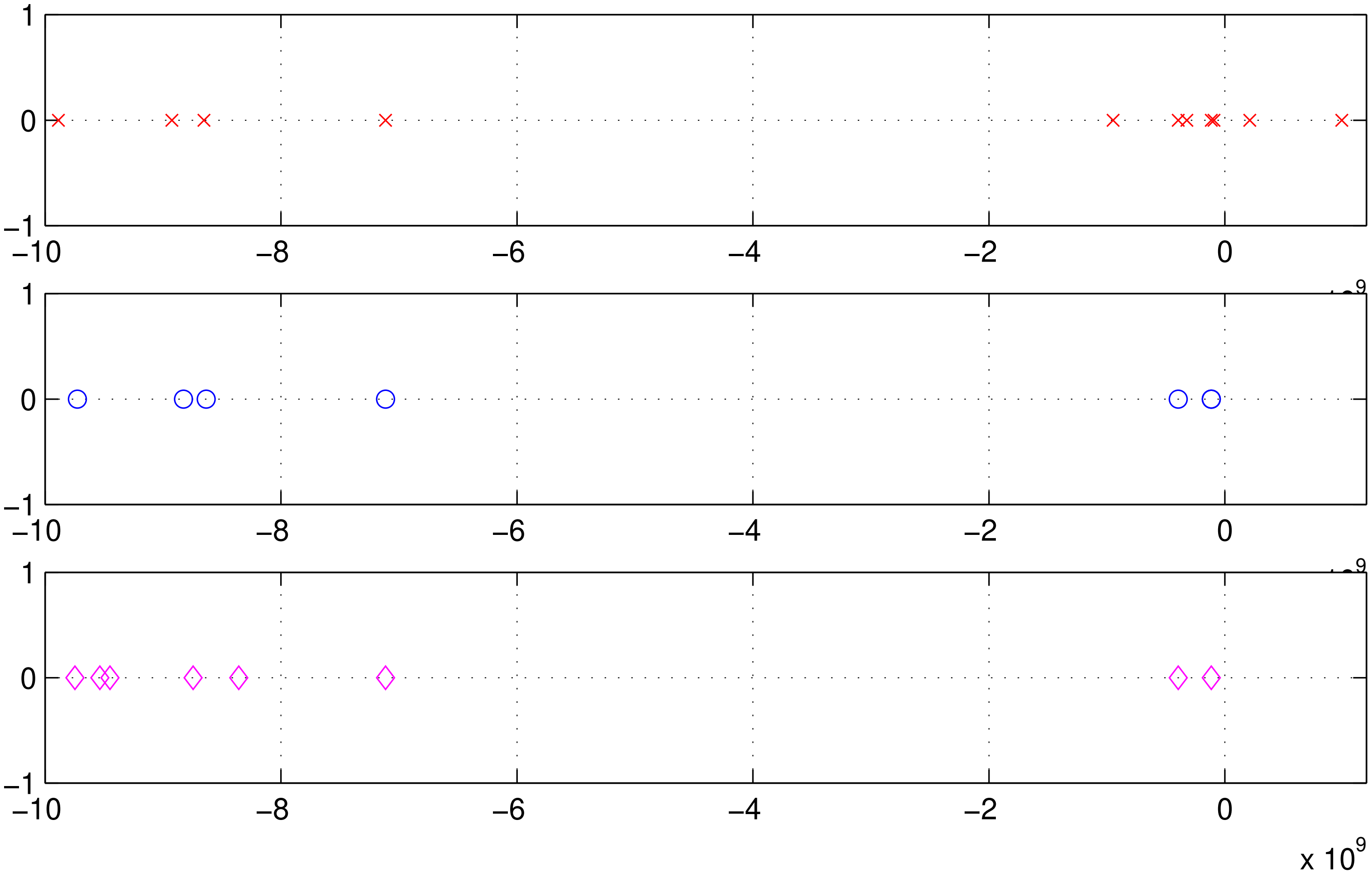
A second example

- $\approx 30,000$ RC elements, $p = 30$ I/O ports
- Reduced-order model of order $n = 60$

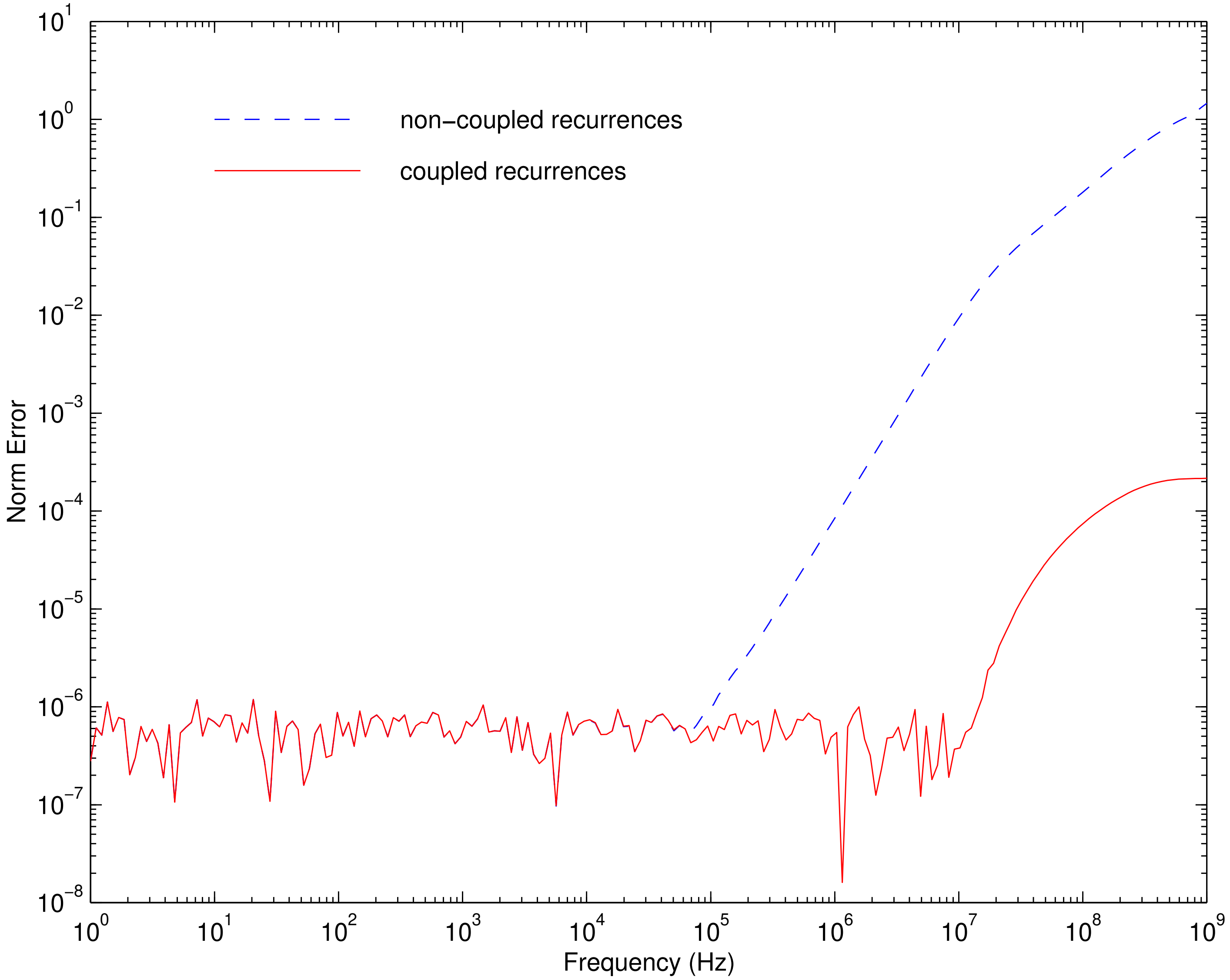
Poles from non-coupled recurrences

Poles from coupled recurrences

Exact poles



Frequency-domain error



Passivity via projection

- Recall: RCL subcircuits are of the form

$$\mathbf{C}\frac{d\,\mathbf{x}}{d\,t} + \mathbf{G}\,\mathbf{x} = \mathbf{B}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{B}^{\mathsf{T}}\,\mathbf{x}(t)$$

where

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{12}^{\mathsf{T}} & \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

and $\mathbf{G}_{11}, \mathbf{C}_{11}, \mathbf{C}_{22}$ are symmetric positive semidefinite

- We may replace $\mathbf{G}$ and $\mathbf{C}$ by

$$\widetilde{\mathbf{G}} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ -\mathbf{G}_{12}^{\mathsf{T}} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix}$$

Passivity via projection, continued

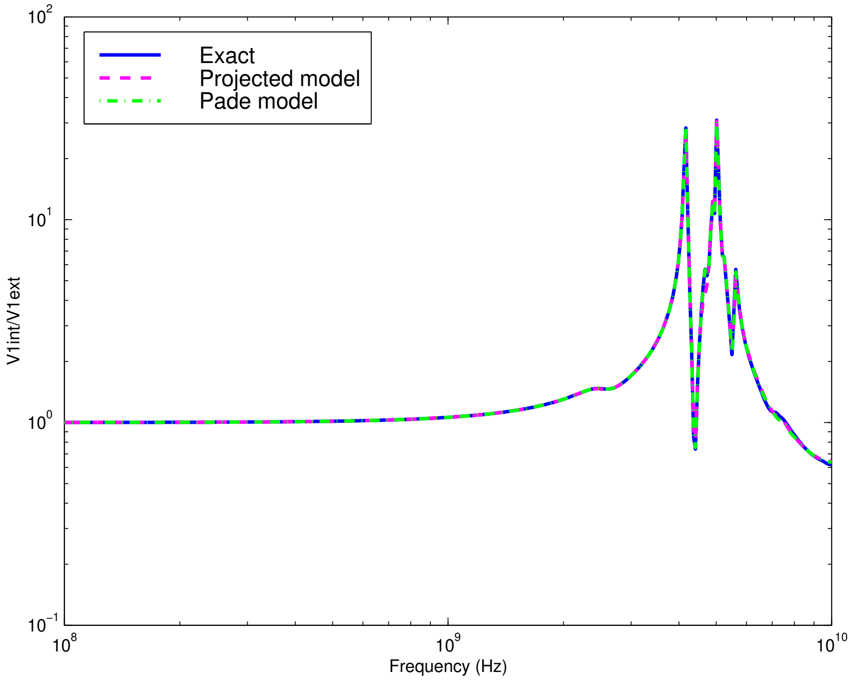
- Run **SyMPVL** (on $\mathbf{G}, \mathbf{C}, \mathbf{B}$) for n steps
- Save all Lanczos vectors: $\mathbf{V}_n$
- Projection of $\widetilde{\mathbf{G}}, \widetilde{\mathbf{C}}, \mathbf{B}$ onto $\mathbf{V}_n$ gives passive model with transfer function $\mathbf{H}_n^{(\mathsf{pas})}$
- Half as accurate as the Padé model:

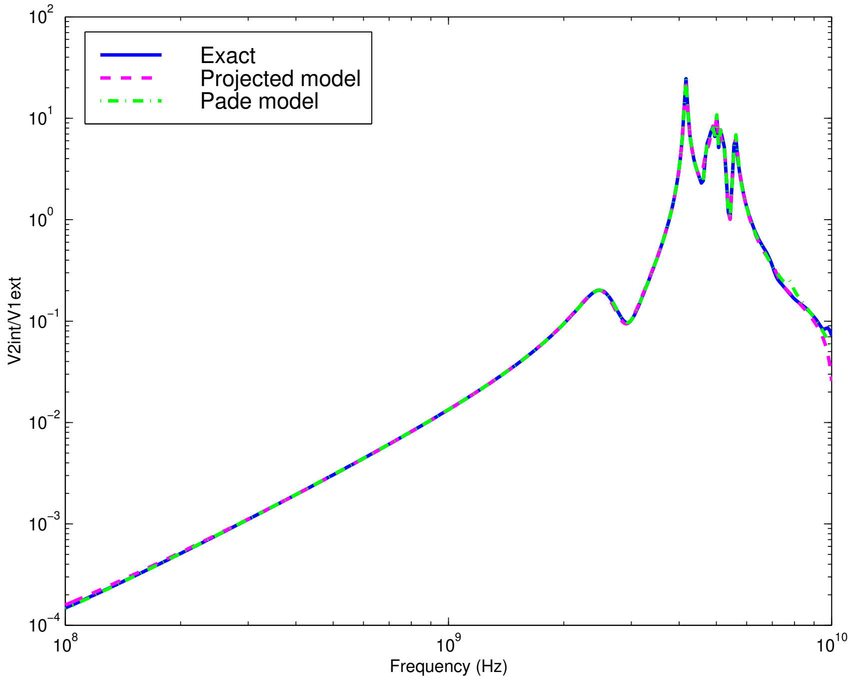
$$\mathbf{H}_n^{(\text{Padé})}(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{q(n)}\right)$$

$$\mathbf{H}_n^{(\text{pas})}(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\lfloor q(n)/2 \rfloor}\right)$$

- **PRIMA** (Odabasioglu,'96, Odabasioglu, Celik, and Pileggi, '97): uses the Arnoldi process to generate the same model $\mathbf{H}_n^{(\text{pas})}$, but it cannot generate $\mathbf{H}_n^{(\text{Padé})}$
- Even so, **PRIMA** has slightly higher computational costs than **SyMPVL**

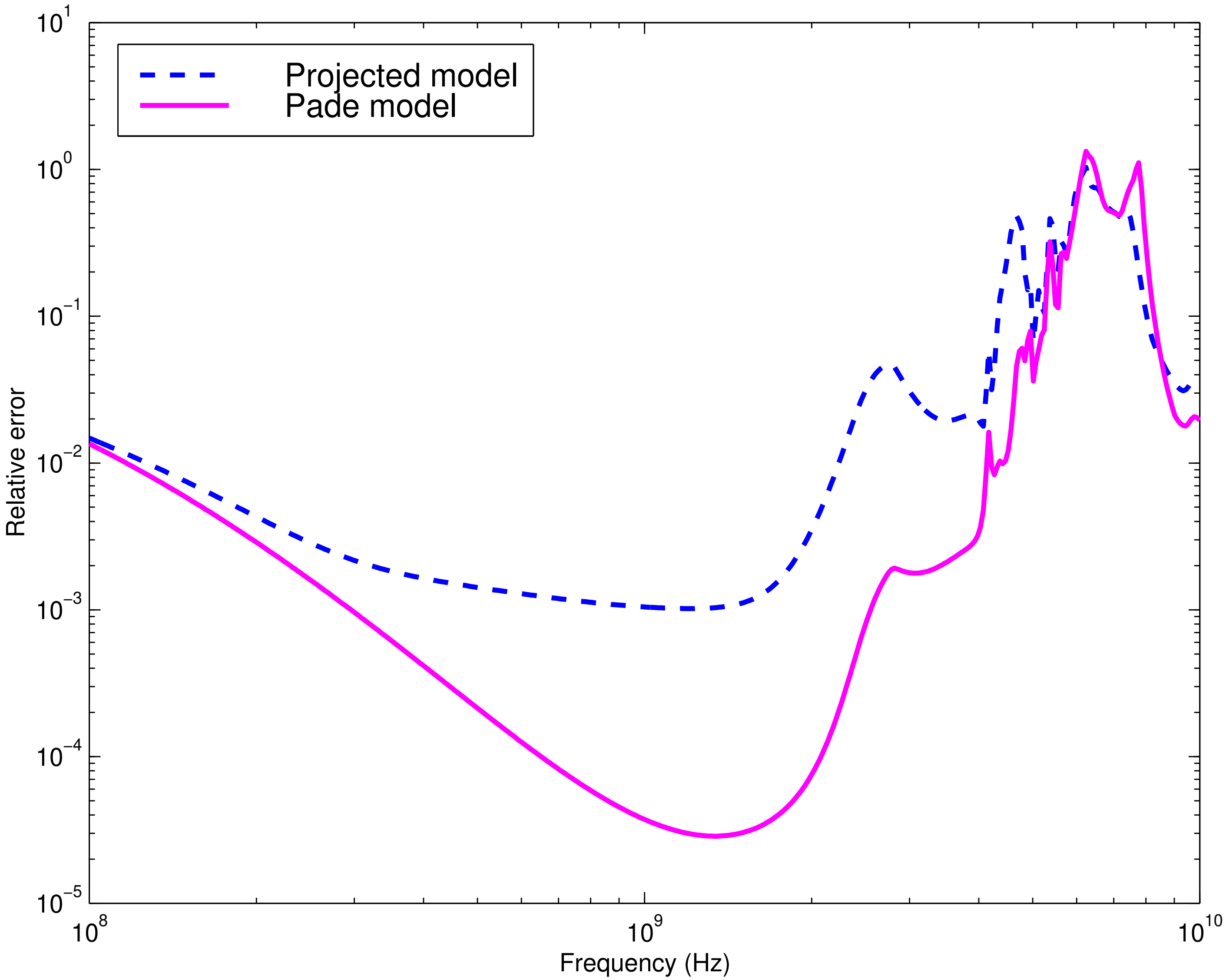
Passive reduced-order model



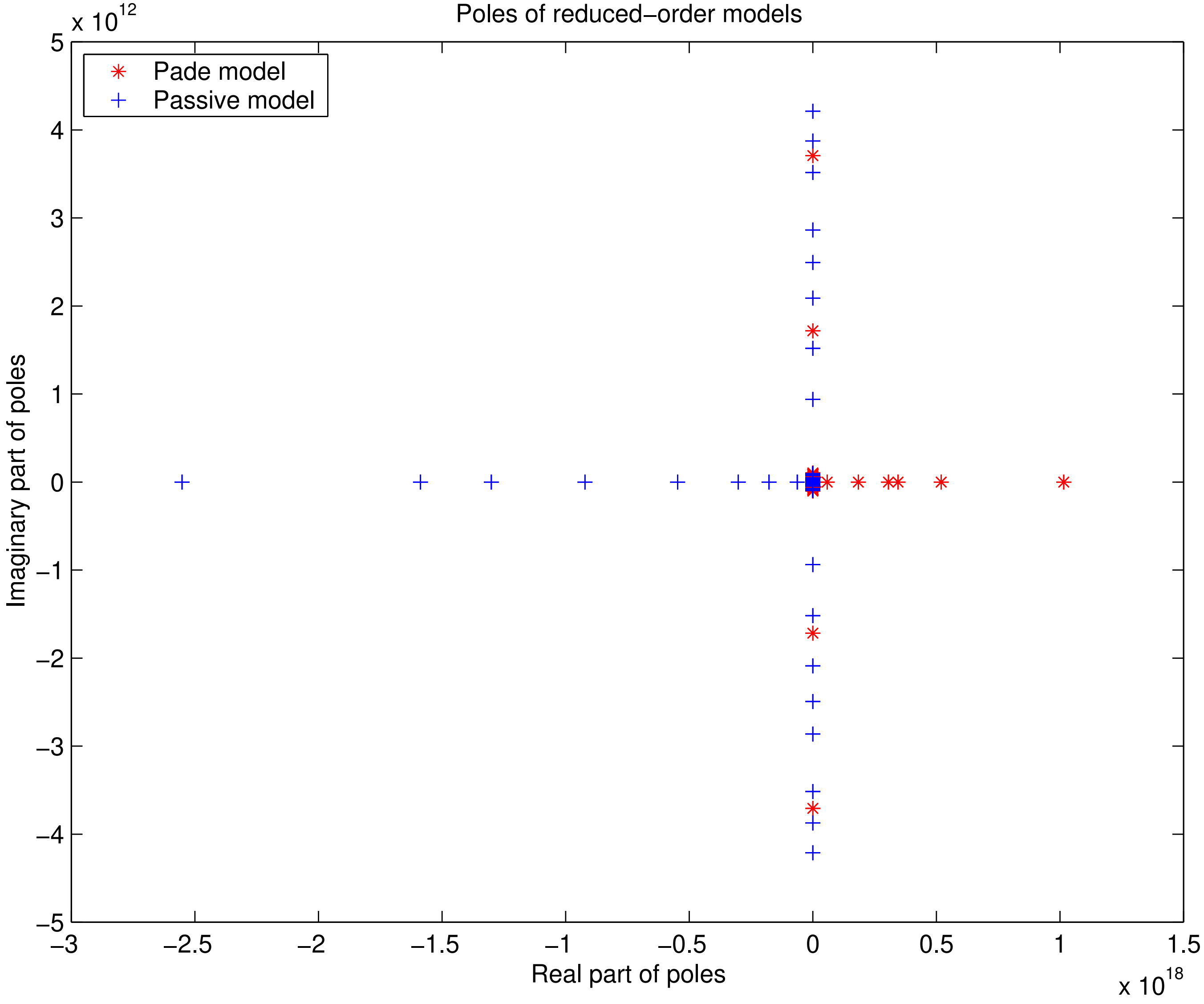


We now need order $n = 112$, instead of $n = 80$

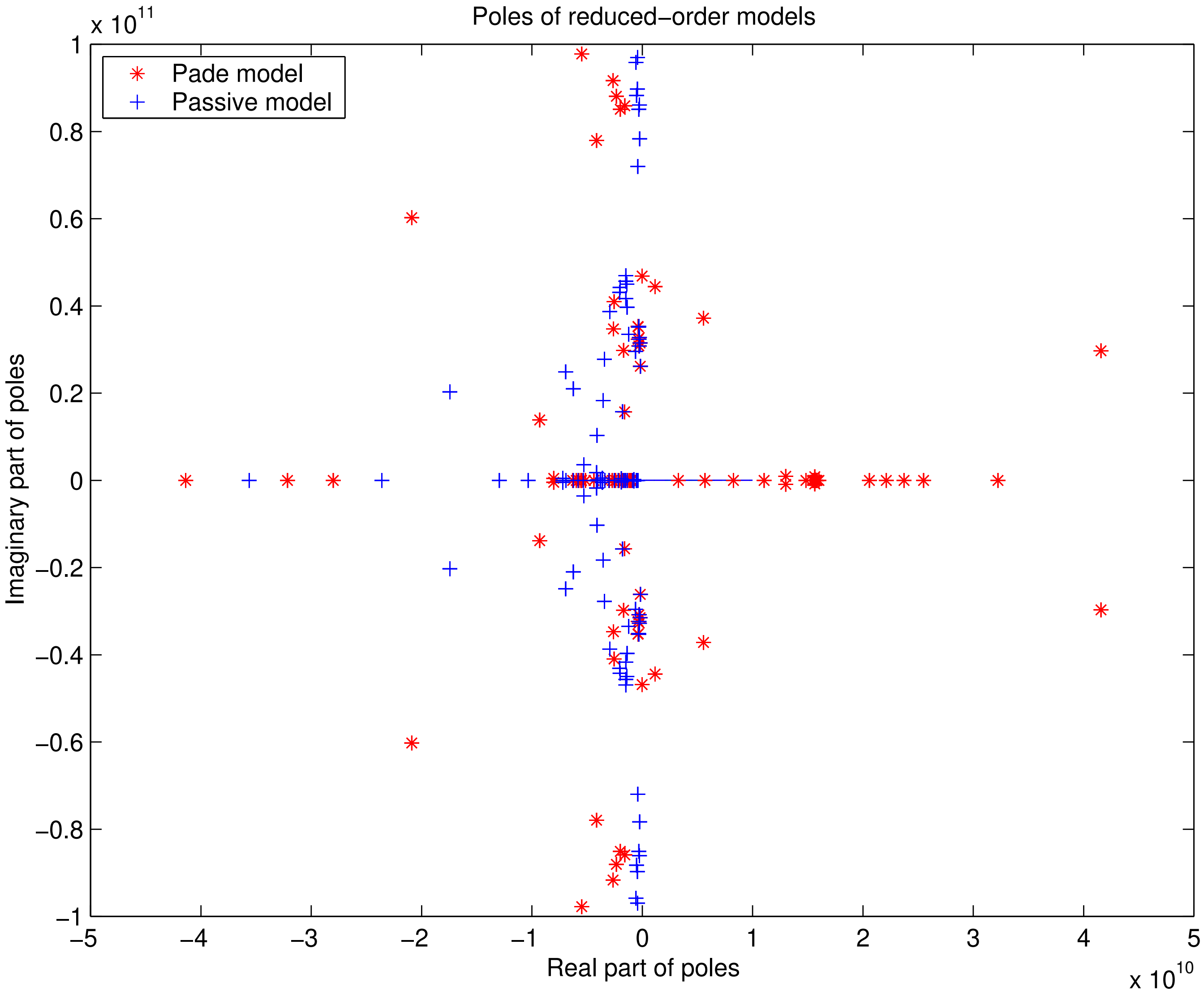
Relative error of both models



Poles of Padé and passive models



Poles near the frequency range of interest



Open problem I

- For general RCL subcircuits, given the size n of the reduced-order model, generate a passive model such that

$$\mathbf{H}_n(s) = \mathbf{H}(s) + \mathcal{O}\left((s - s_0)^{\tilde{q}}\right)$$

with maximal possible approximation order $\tilde{q} = \tilde{q}(n)$

- Projection gives models with

$$\tilde{q}(n) = \lfloor n/p \rfloor$$

- Often, the optimal order is closer to

$$2\lfloor n/p \rfloor$$

- For RC, RL, and LC subcircuits, we have

$$\tilde{q}(n) \geq 2\lfloor n/p \rfloor$$

Open problem II

- How do we check if a given model is passive?
- "No poles with positive real part" is only necessary, but not sufficient
- Positive real lemma for descriptor systems (F. and Jarre, '01):

$$\mathbf{H}(s) = \mathbf{B}^\top (\mathbf{G} + s\,\mathbf{C})^{-1}\,\mathbf{B}$$

  is positive real "iff" the LMIs

$$\mathbf{G}^\top \mathbf{X} + \mathbf{X}^\top \mathbf{G} \geq \mathbf{0}, \quad \mathbf{C}^\top \mathbf{X} = \mathbf{X}^\top \mathbf{C} \geq \mathbf{0}, \quad \mathbf{X}^\top \mathbf{B} = \mathbf{B}$$

  have a solution $\mathbf{X} \in \mathbb{R}^{n \times n}$

- Need to solve bilinear semidefinite program with $\mathcal{O}(n^2)$ variables
- Only possible for very small n

Concluding remarks

- Reduced-order modeling has become crucial tool in circuit simulation
- Matrix-Padé models can be generated efficiently via a Lanczos-type method
- Matrix-Padé models of passive systems are not passive and not even stable in general
- Passive models of RC, RL, and LC subcircuits via coupled recurrences
- Passive models of subcircuits via projection
- Passive models with optimal approximation properties ?
- Survey paper in Numerical Analysis 2000 issues of JCAM
  Available also from `http://cm.bell-labs.com/who/freund/`