Fundamental extraction algorithms

- **Field equations versus circuit equations**
- **Transmission line case study**
- **Capacitance computation**
- **Resistance computation**
- **Numerical methods used by field solvers**
  - **finite differences**
  - **finite elements**
  - **integral equation techniques**
    - **multipole acceleration**
    - **SVD-based matrix compression**
  - **random-walk**
- **Inductance computation methods**

Interconnect obeys Maxwell's equations

$$\nabla \times E = -\frac{\partial B}{\partial t}, \qquad \nabla \cdot B = 0$$
$$\nabla \times H = \frac{\partial D}{\partial t} + J, \quad \nabla \cdot D = \rho$$

or

$$\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS, \qquad \int_S B \cdot dS = 0$$
$$\oint H \cdot dl = \int_S J \cdot dS + \int_S \frac{\partial D}{\partial t} \cdot dS, \quad \int_S D \cdot dS = \int_V \rho dV$$

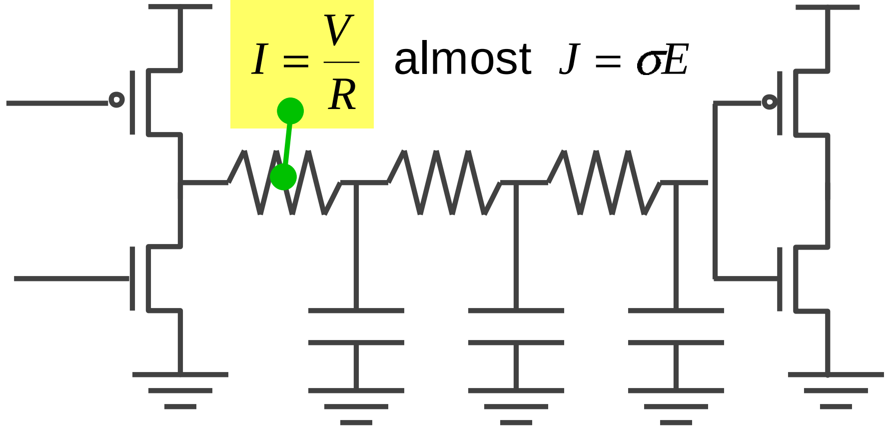
Auxiliary medium - dependent equations

$J = \sigma E$ - Ohm's law

$B = \varepsilon E, \qquad B = \mu H$

in homogeneous, isotropic, linear, time - invariant media

Designers prefer Kirchhoff



$I = \frac{V}{R}$ almost $J = \sigma E$

KCL: $\sum_{i \in S} I_i = 0$ consequence of $\oint H \cdot dl = \int_S J \cdot dS + \int_S \frac{\partial D}{\partial t} \cdot dS$ and $\int_S D \cdot dS = \int_V \rho dV$

,

KVL: $\sum_{i \in C} V_i = 0$ consequence of $\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS$

The lumped-element circuit abstraction

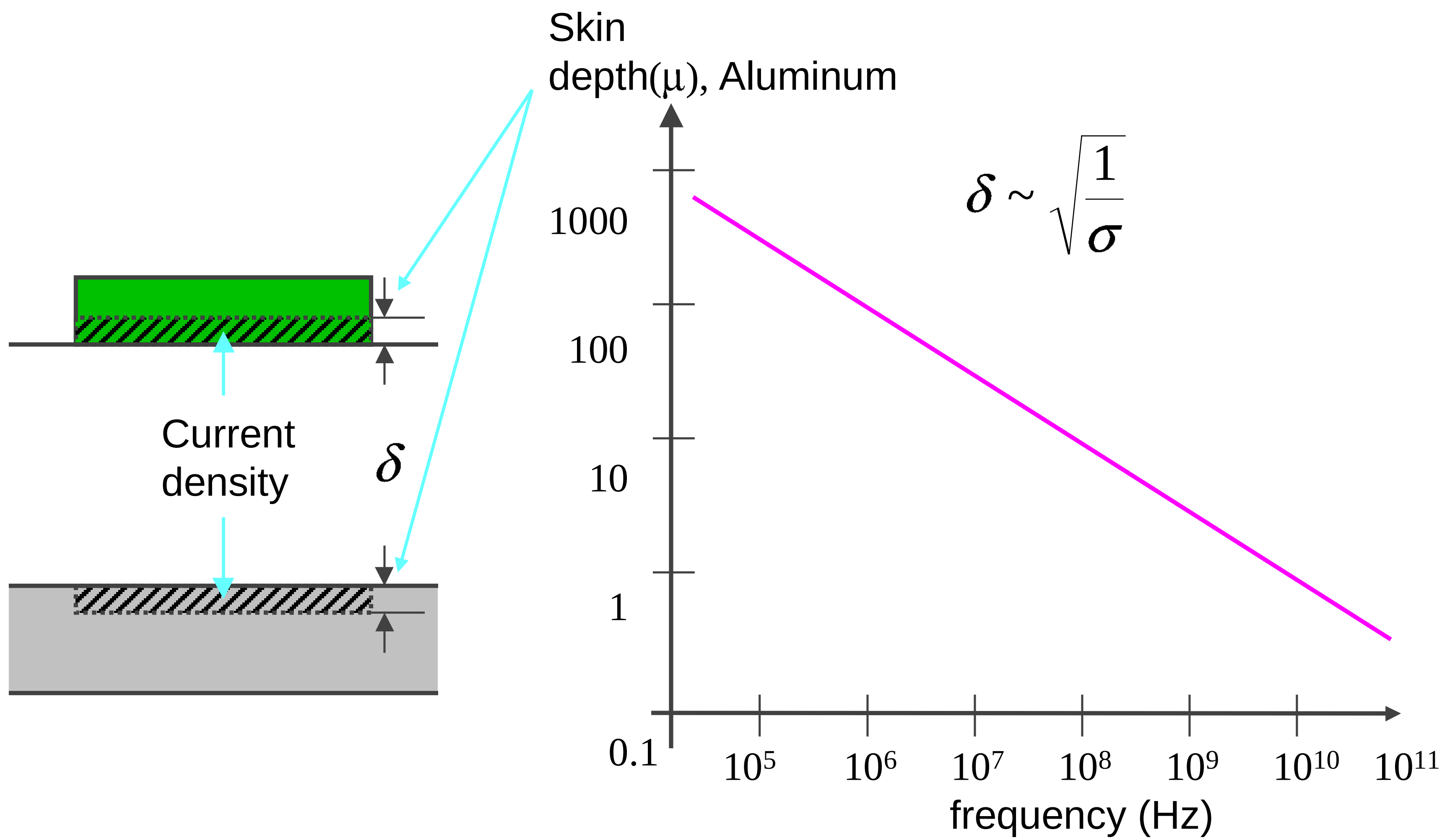
- **Energy storage elements - capacitors/inductors,**
- **Dissipative elements - resistors,**
- **Mutual couplings, electrical and magnetic, made explicit.**

- **Advantages:**
  - — **well separated functions,**
  - — **better understanding of cause and effect,**
  - — **powerfull analysis and synthesis methods.**

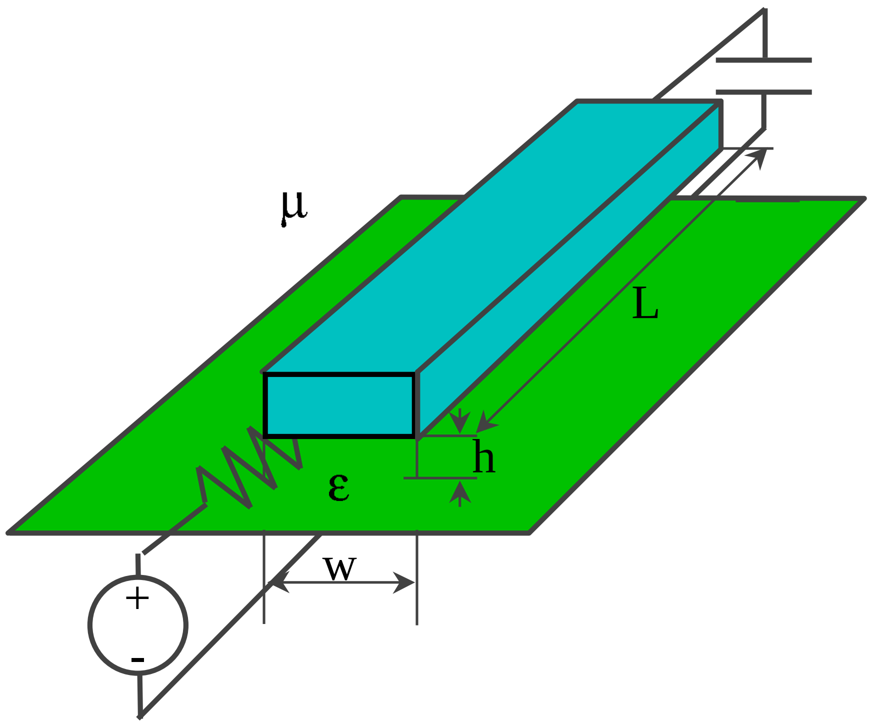
- **Elements small compared to the wavelength** ($\sim$1cm = 10^8 X 10^{-10})
  - — **Fields of elemens are *quasistatic***
    (time-varying, with spatial form of a static field).

- **Distributed effects, e.g., transmission-lines**
  - — **can sometimes be modeled by lumped-elements**

Skin effect

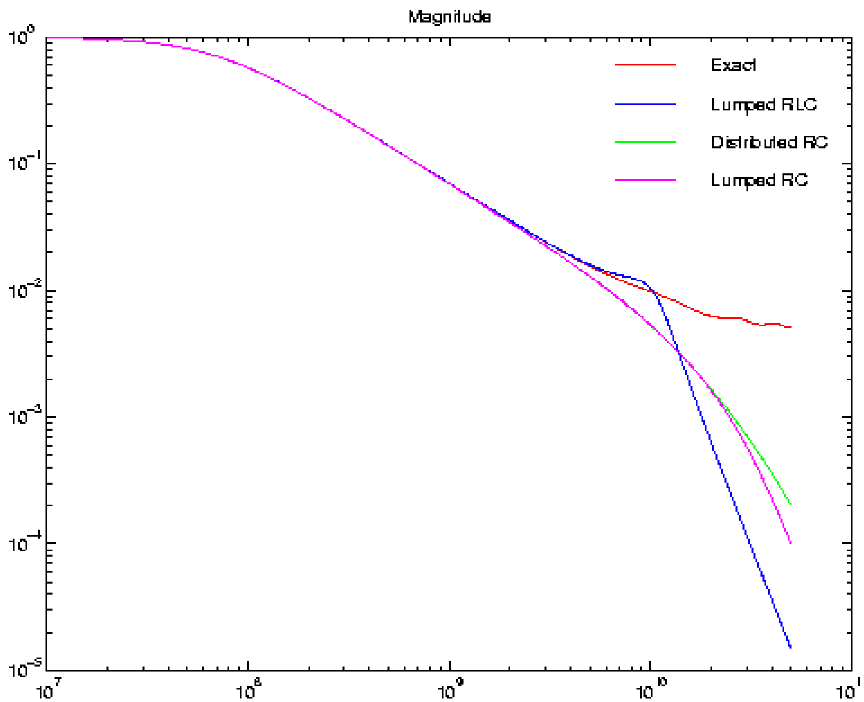

Skin
depth(μ), Aluminum
1000
100
10
1
0.1
$\delta \sim \sqrt{\frac{1}{\sigma}}$
10^5
10^6
10^7
10^8
10^9
10^{10}
10^{11}
frequency (Hz)
Current
density
δ

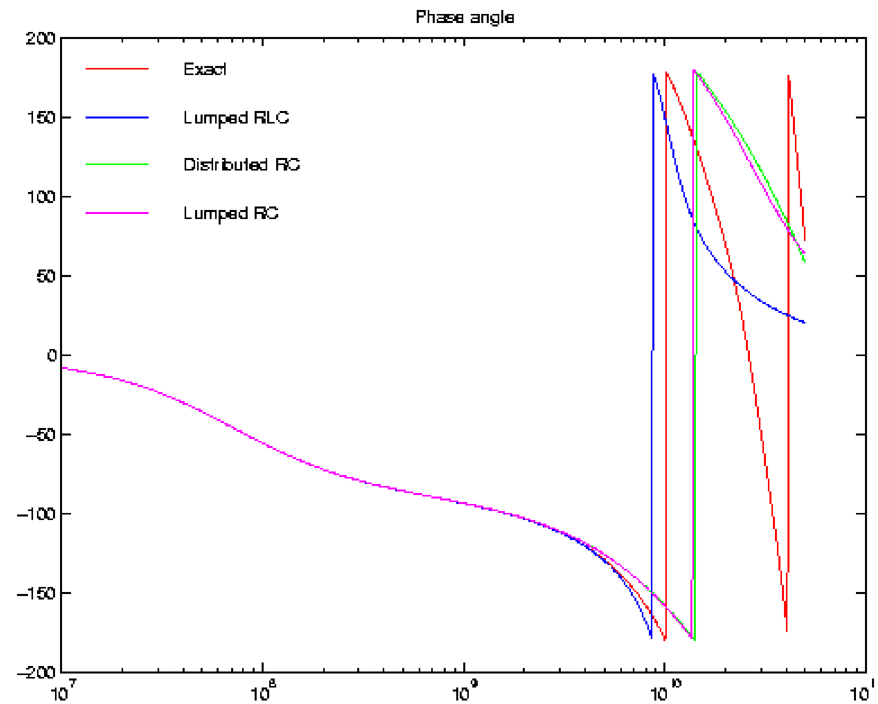
Analysis of a transmission line



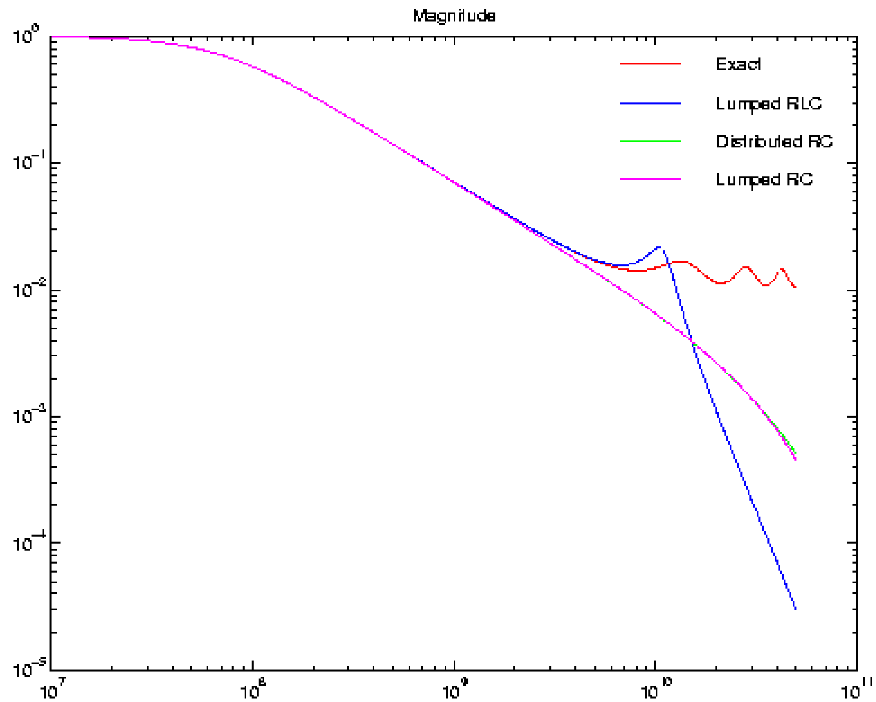
- width w=50μ
- height h=1μ
- vary length L
- study effect of lowering ε (low K dielectrics)
- study effect of increasing σ (Copper interconnect)

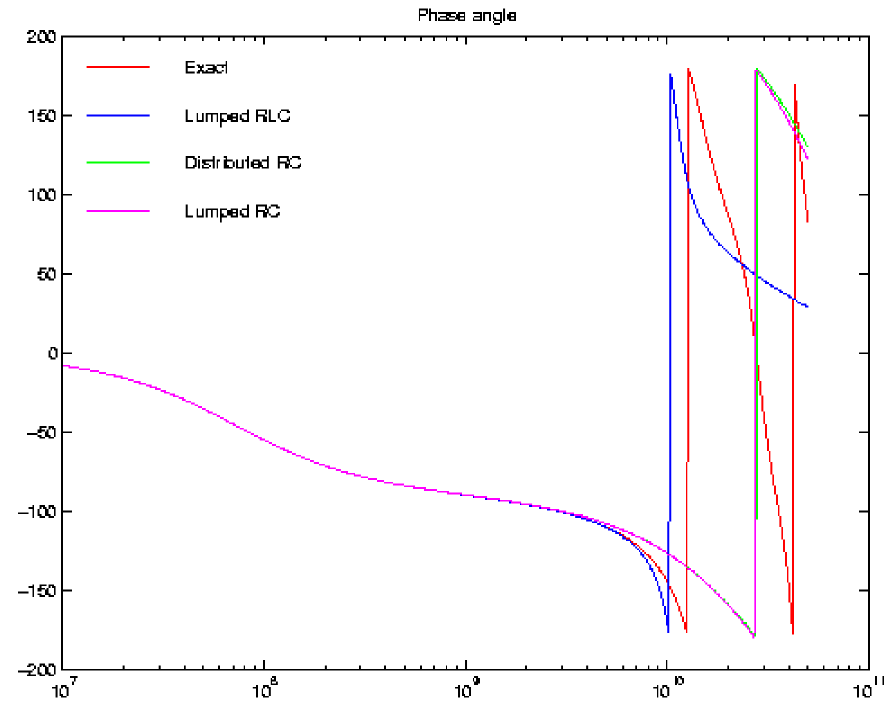
Nominal 10mm wire



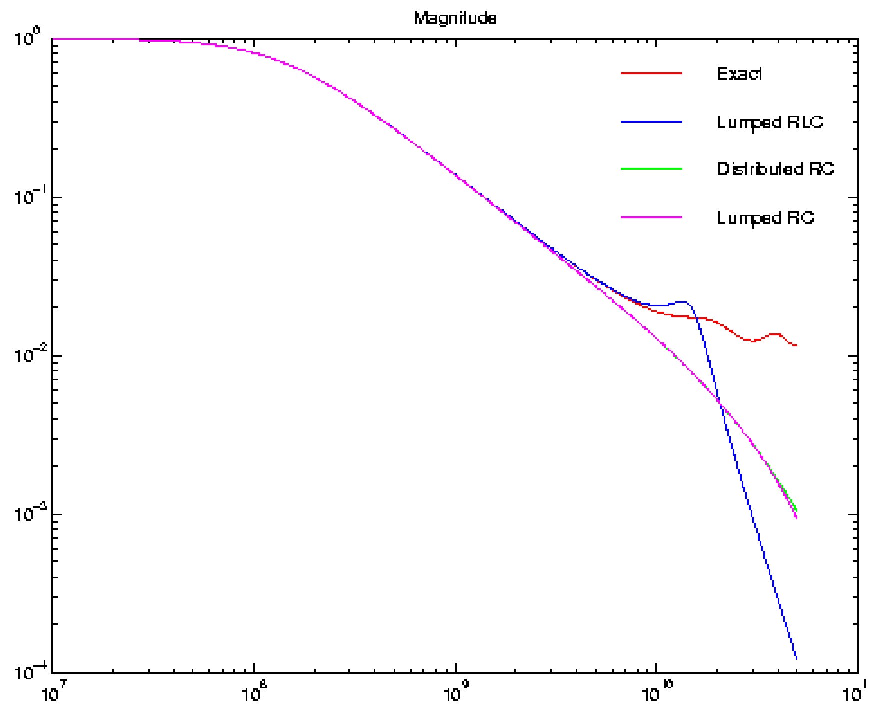


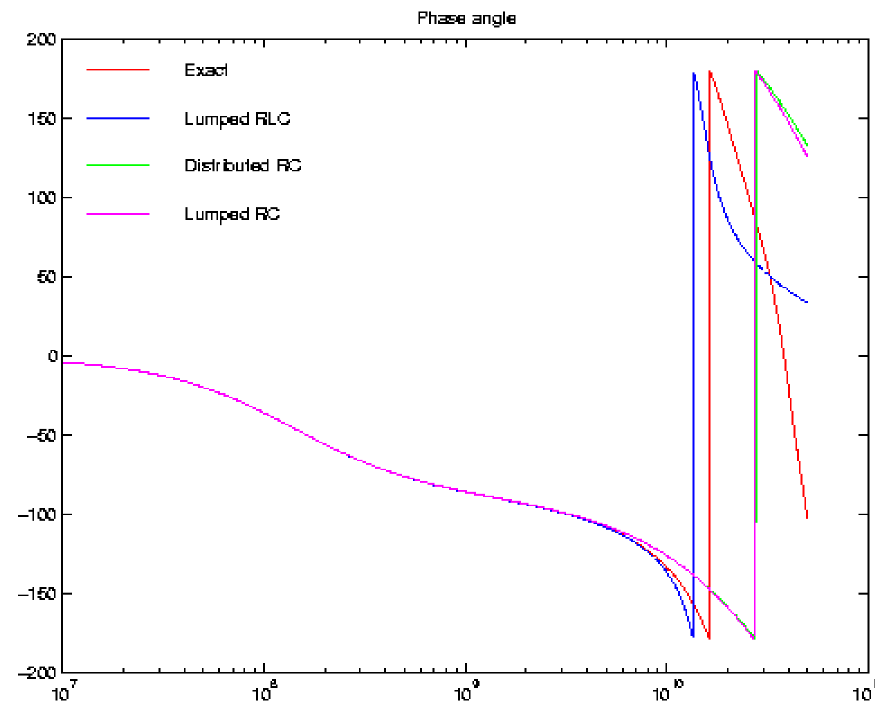
10 mm wire, low resistivity



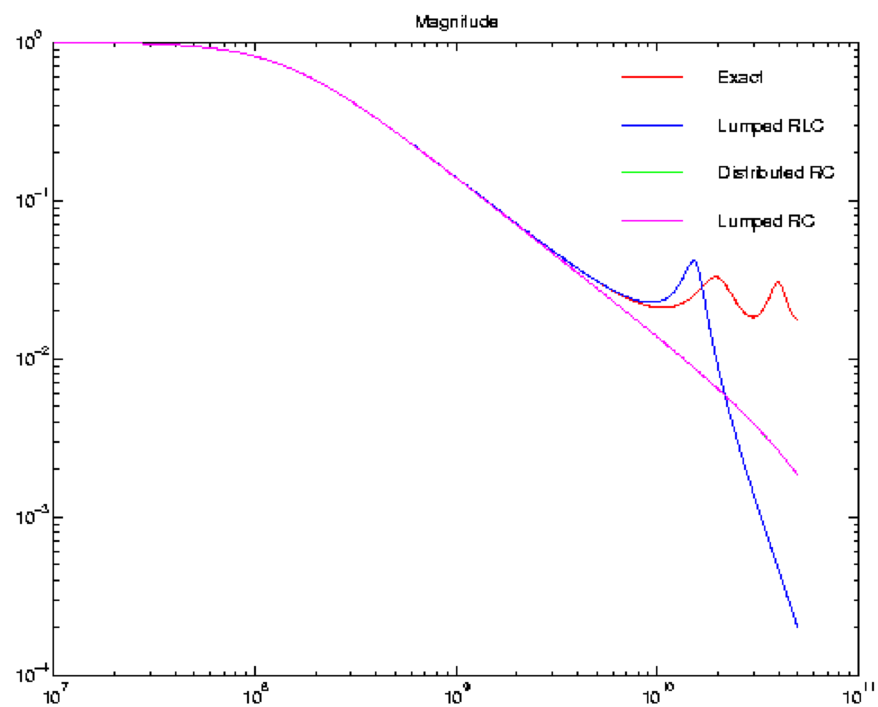


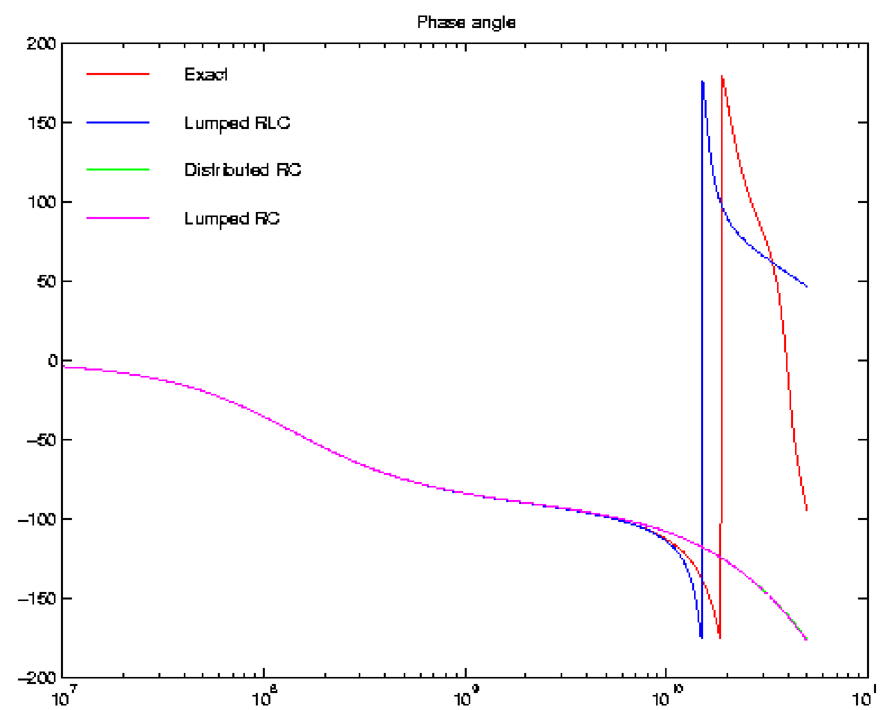
10 mm wire, low K dielectric



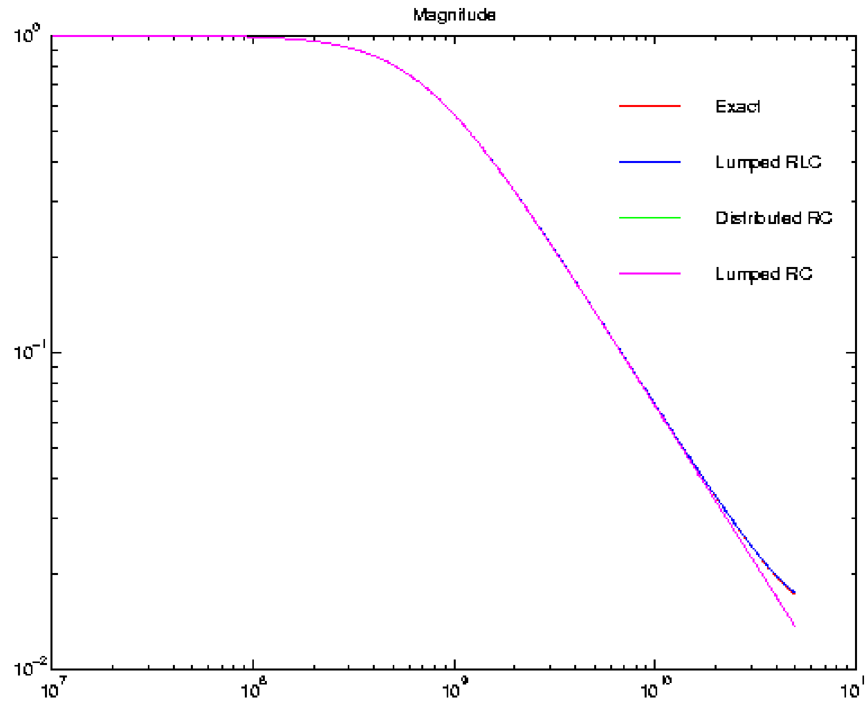


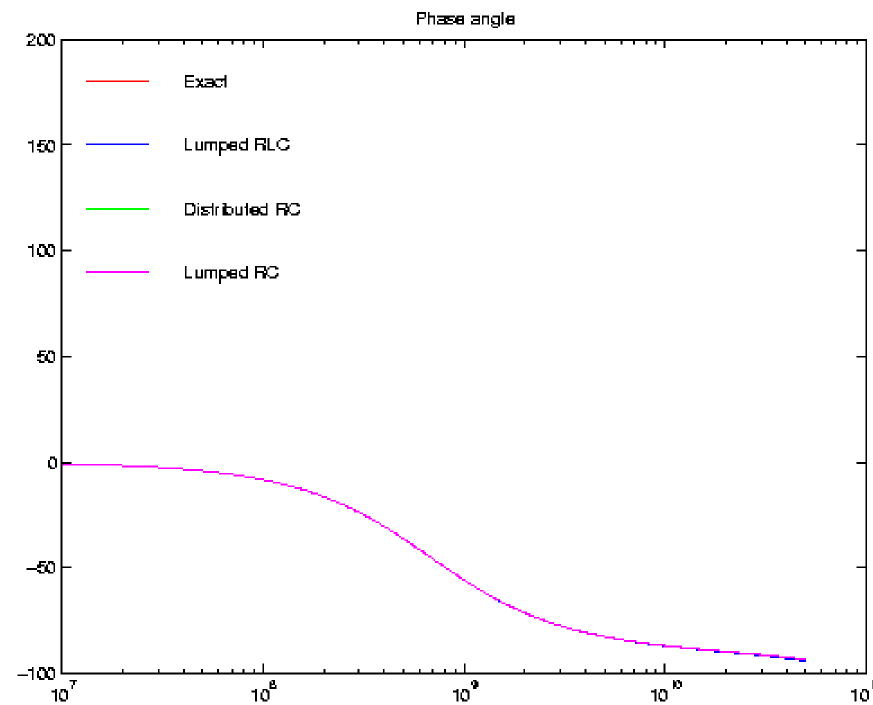
10mm wire, low K and low resistivity



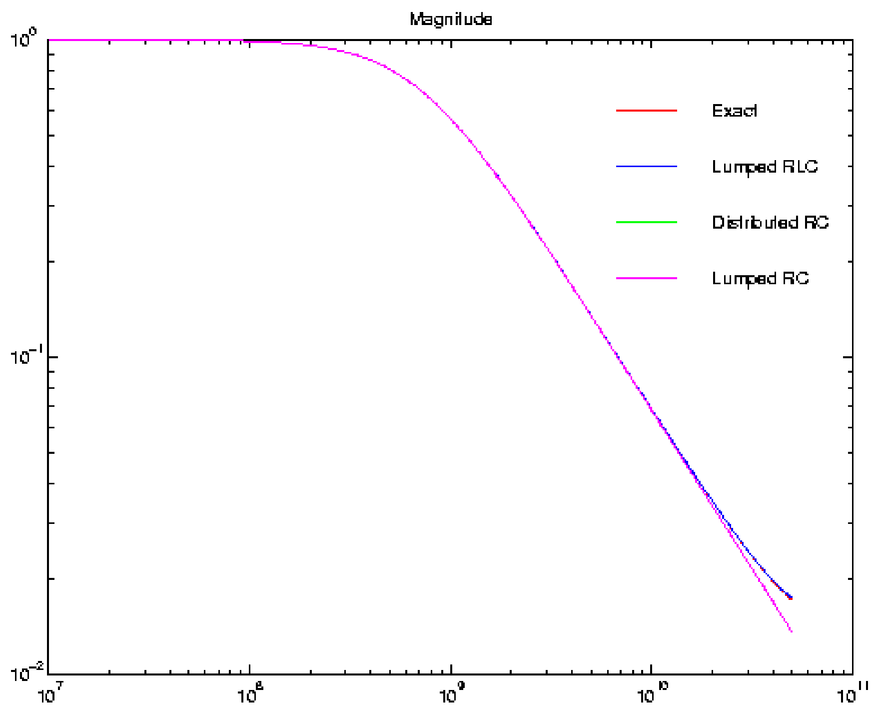


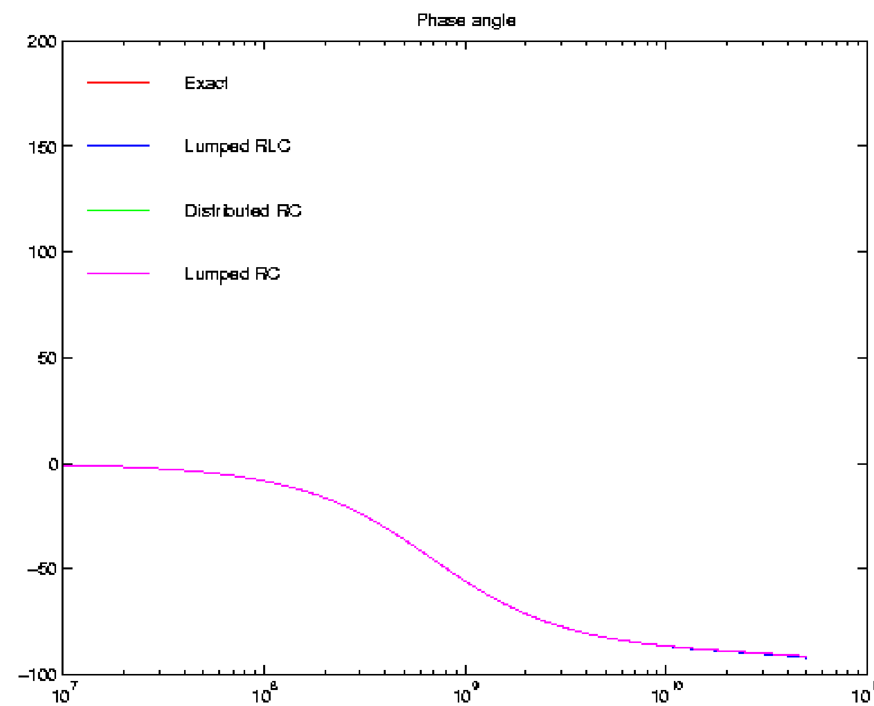
Nominal 1mm line





1 mm line, low resistivity

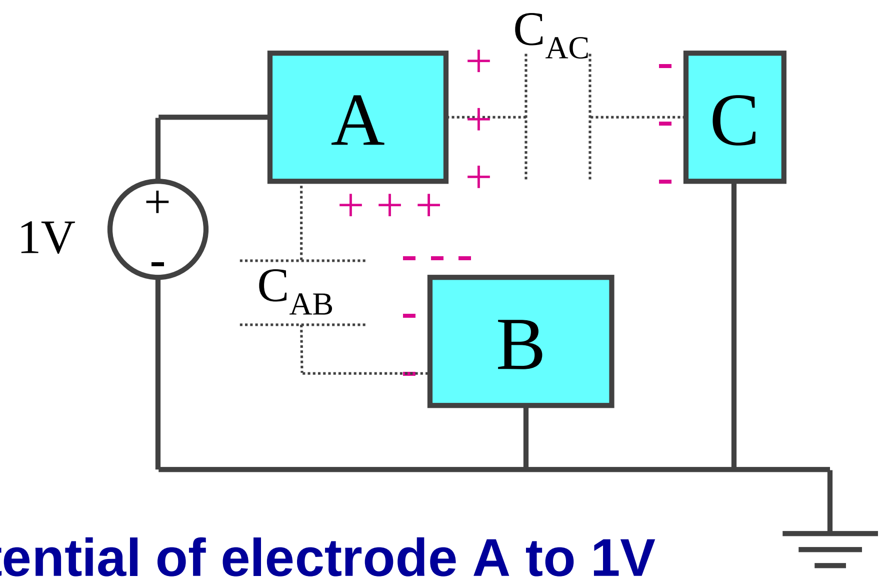




Discussion

- Inductance matters only for global wires in the practical frequency range.
- Similarly, distributed effects are important only in global wires.
- It is difficult to model global wires by lumped RLC models.
- In “shorter” wires only RC effects matter.
- There is no need for distributed RC modeling.
- Copper wires exhibit comparatively higher inductive effects.
- Low-K dielectrics have the opposite effect.

Capacitance computation



- Set potential of electrode A to 1V
- Ground all other electrodes
- Compute the induced charges on the grounded electrodes

$$C_{AB} = -\int_B \rho dv, \quad C_{AC} = -\int_C \rho dv$$

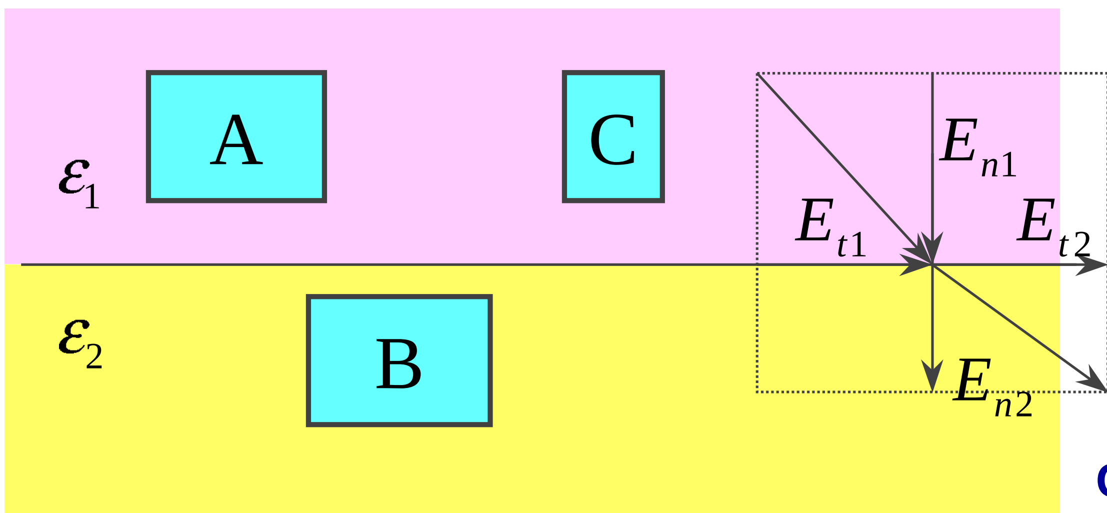
- Repeat for the other electrodes

Capacitance extraction

- **Solve Poisson's equation**

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon}$$

- **Boundary conditions**



$$E_{t1} = E_{t2}$$

$$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

$$D_{n1} - D_{n2} = \rho_s$$

Conditions on derivative:

$$E = -\nabla\Phi$$

Resistance calculation

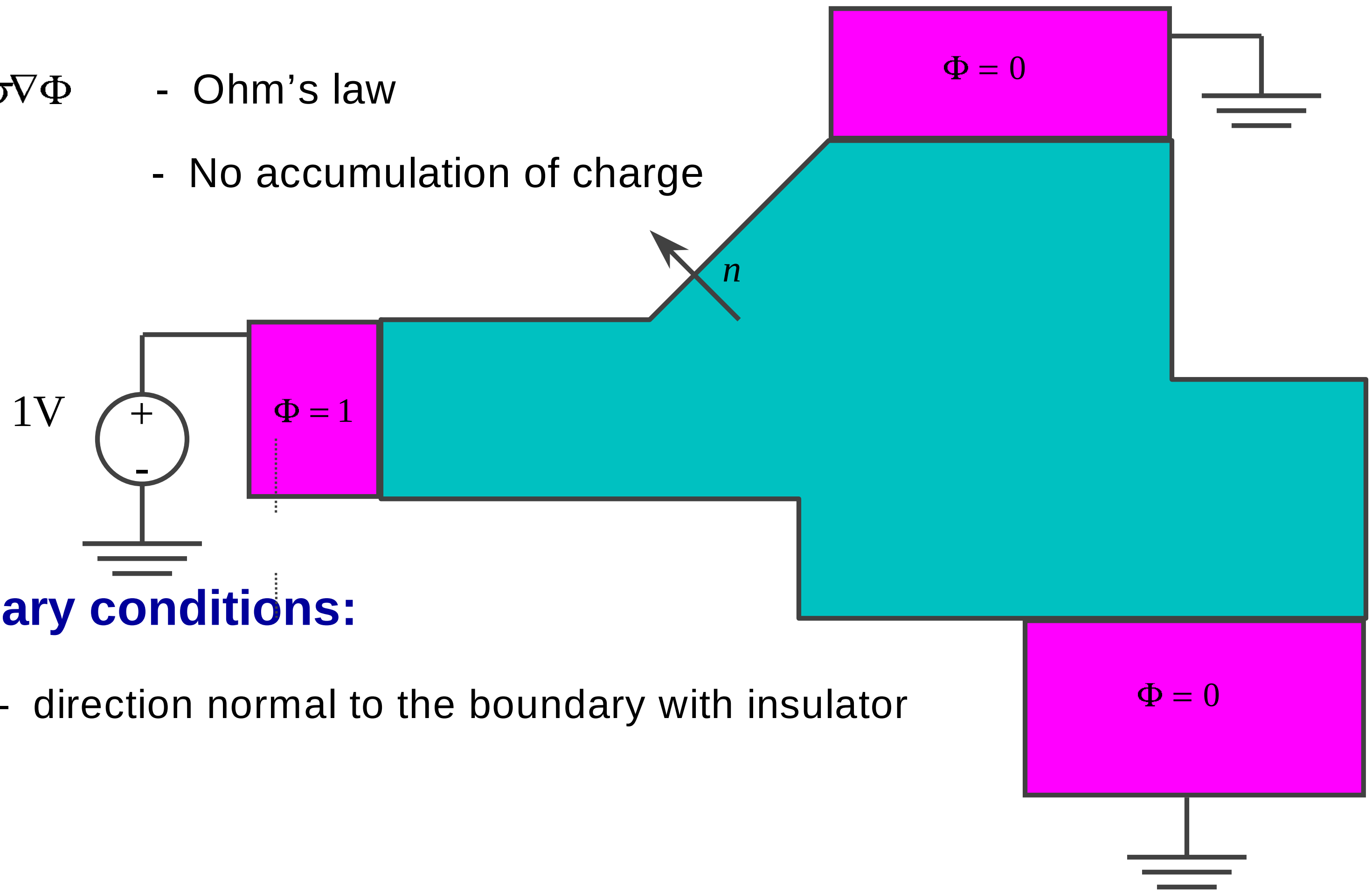
- **Quasi-static assumption => Laplace's equation:**

$J = \sigma E = -\sigma \nabla \Phi$ - Ohm's law

$\nabla \cdot J = 0$ - No accumulation of charge

results:

$\nabla^2 \Phi = 0$



- **Boundary conditions:**

$\frac{\partial \Phi}{\partial n} = 0,$ n - direction normal to the boundary with insulator

The finite-difference method

- **Approximate derivative** $\frac{\partial \Phi}{\partial x}$
  - **e.g. central difference formula - error** $O(h^2)$

$$\frac{d\Phi}{dx} \cong \frac{\Phi(x+h) - \Phi(x-h)}{2h}$$

- **second order derivative** $\frac{\partial^2 \Phi}{\partial x^2}$**:**

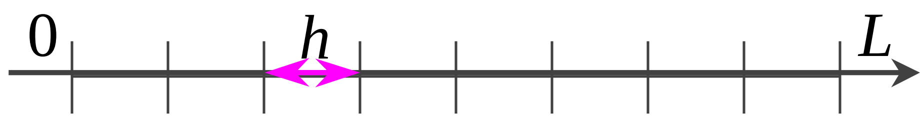
$$\frac{\partial^2 \Phi}{\partial x^2} \cong \frac{\Phi(x+h) - 2\Phi(x) + \Phi(x-h)}{h^2}$$ **- error** $O(h^2)$

- **At boundary:**
  - **use forward or backward difference** $O(h)$
  - **introduce additional dummy point to preserve order.**

Example: 1-dim Laplace equation

$$\frac{d^2\Phi}{dx^2} = 0, \qquad \Phi(0) = \Phi_l, \quad \Phi(L) = \Phi_r$$

- substitute approximate derivative formula



$$\frac{d^2\Phi}{dx^2} \cong \frac{\Phi(x+h) - 2\Phi(x) + \Phi(x-h)}{h^2}$$

- Large, very sparse system of linear equations:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ 0 \\ \vdots \\ 0 \\ \Phi_n \end{bmatrix} = \begin{bmatrix} \Phi_l \\ 0 \\ \vdots \\ 0 \\ \Phi_r \end{bmatrix}$$

The finite-difference method (cont.)

- **Grid can be non-equidistant.**
- **Boundary conditions:**
  - — **specify applied** Φ
  - — **specify conditions involving** $E_x = -\frac{d\Phi}{dx}$ **on boundary.**
- **Problems in 2 or 3 dimensions.**
- **Must discretize entire space (in theory to infinity) or use equivalent boundary conditions.**
- **Leads to large, very sparse system of linear equations.**
- **Solved by direct sparse factorization or by iterative linear system solution methods.**

The finite-element method

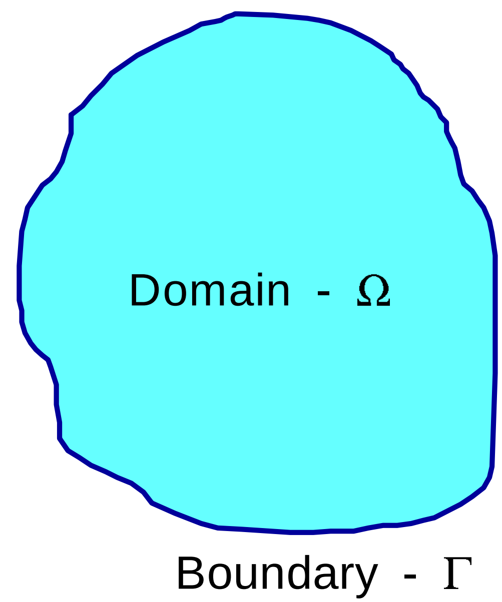
- **Finite differences => approximate Φ at a set of points**
- **Generalization - approximate function:**

$$\Phi \cong \hat{\Phi} = \psi + \sum_{n=1}^{M} a_n \varphi_n$$

$\varphi_n, n = 1, \cdots, M$ - trial functions

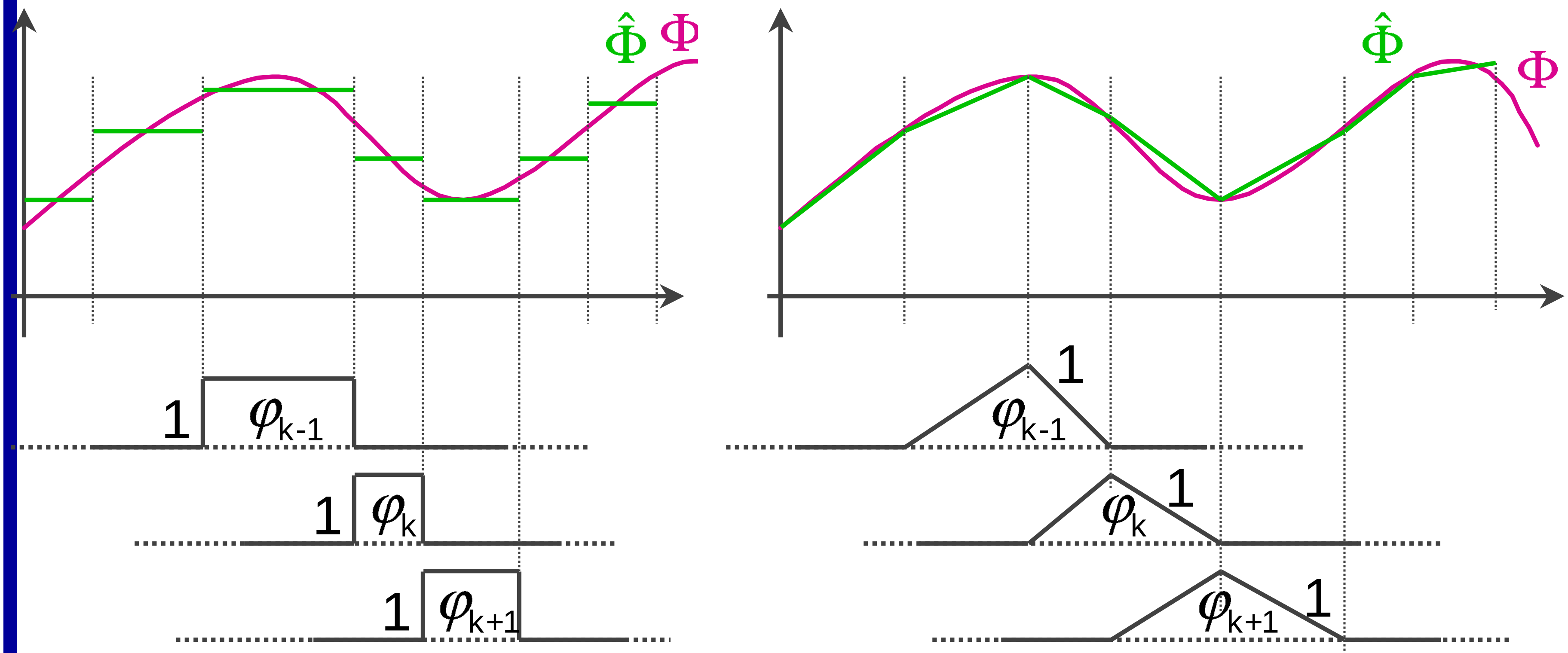
$a_n, n = 1, \cdots, M$ - coefficients

$\psi|_\Gamma = \Phi|_\Gamma$ - match on boundary



Example 1-dim test functions

- **Piecewise defined functions**



- **Generalizes to higher orders and dimensions**

The finite-element discretization

- **Weighted residual approximations (weight functions w_l):**

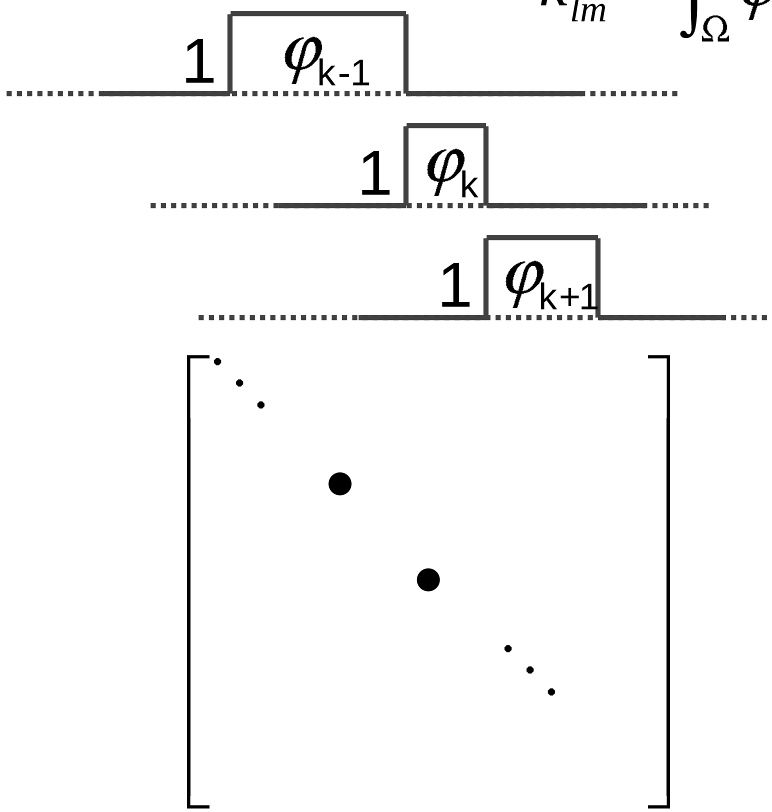
$$\int_\Omega w_l\left(\Phi-\hat{\Phi}\right)d\omega=0,\quad n=1,\cdots,M$$

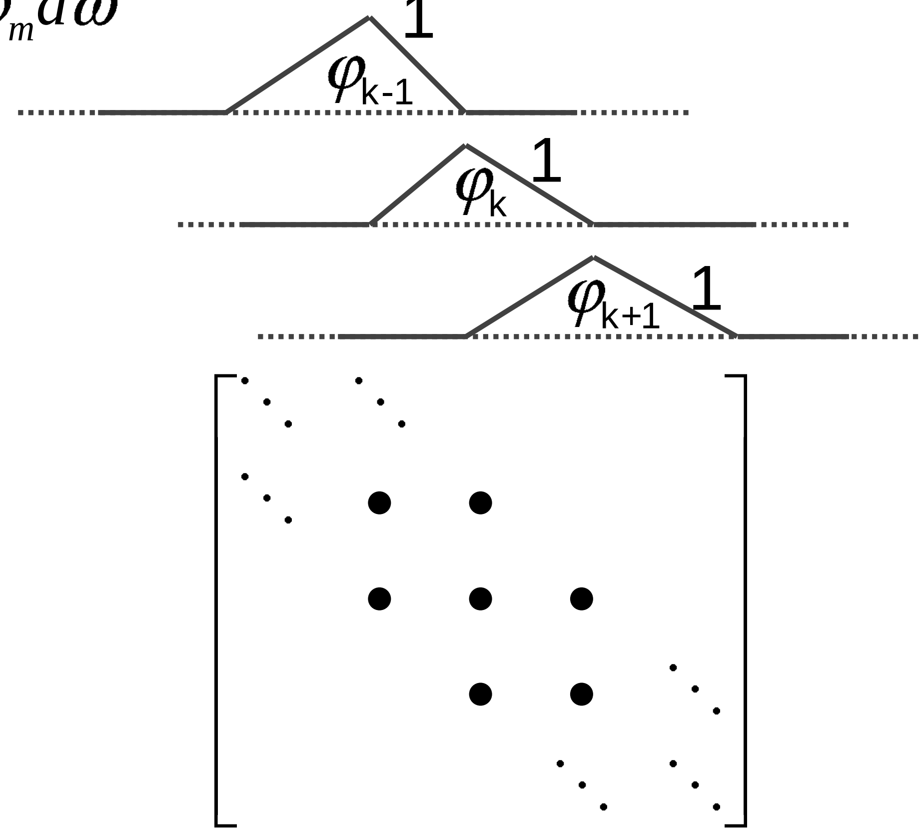
- **Results in linear system of equations:** $\mathbf{Ka}=\mathbf{f}$
  - **coefficients:** a_l
  - **matrix elements:** $k_{lm}=\int_\Omega w_l\varphi_m d\omega$
  - **right hand side:** $f_l=\int_\Omega w_l(\varphi-\psi)d\omega$
- **Choice of weight functions:**
  - **point collocation:** $w_l=\delta(x-x_l)$
  - **Galerkin:** $w_l=\varphi_l$

The finite-element method(cont.)

- **Resulting discretized matrix is sparse (banded):**

$$k_{lm} = \int_{\Omega} \varphi_l \varphi_m d\omega$$





- **Trade-off: Higher order => wider bandwidth**
  Higher order => smaller matrix (for same accuracy).

Example: 1-dim equation

$$\frac{d^2\Phi}{dx^2} - \Phi = 0, \qquad \Phi(0) = \Phi_l, \quad \Phi(L) = \Phi_r$$

- **Weighted residual statement:**

$$\hat{\Phi} = \sum_{m=1}^{M+1} a_m \varphi_m, \qquad m = 1, \cdots, M+1$$

$$\int_0^L w_l \left(\frac{d^2\hat{\Phi}}{dx^2} - \hat{\Phi} \right) dx = 0, \quad l = 1, \cdots, M+1$$

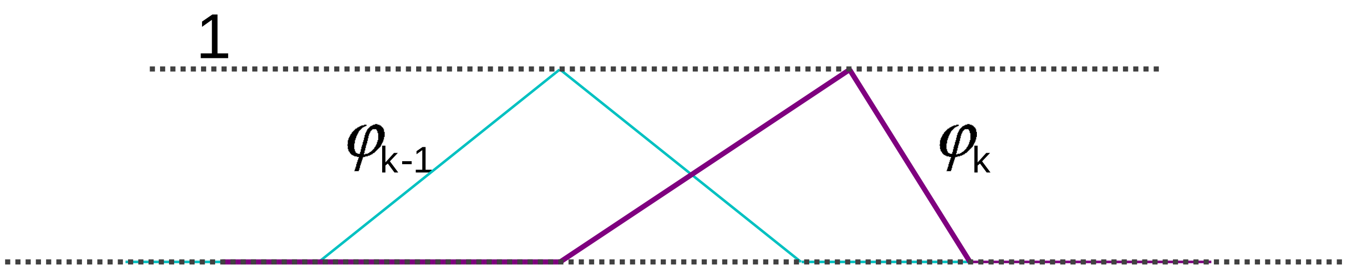
- **Integration by parts:**

$$-\int_0^L \left(\frac{dw_l}{dx} \frac{d\hat{\Phi}}{dx} + w_l \hat{\Phi} \right) dx + \left[w_l \frac{d\hat{\Phi}}{dx} \right]_0^L = 0, \quad l = 1, \cdots, M+1$$

- **Only C^0 continuity is necessary for weight and test functions**

Example (cont.)

$$\mathbf{Ka} = \mathbf{f}$$



$$k_{lm} = \int_0^L \left(\frac{d\varphi_l}{dx} \frac{d\varphi_m}{dx} + \varphi_l \varphi_m \right) dx, \quad 1 \le l, m \le M+1$$

$$f_l = \left[\varphi_l \frac{d\hat{\Phi}}{dx} \right]_0^L, \quad 1 \le l \le M+1$$

- **Simple to calculate!**
- **Tri-diagonal matrix**

Finite-differences/elements

1. **Volume meshing of structures**
2. **Very large sparse matrices**
3. **Matrices are poorly conditioned**
4. **For exterior problems, hard to enforce boundary conditions**
5. **Computationally very expensive**

- **Used when dielectric variation is complex**

- **Commercial implementations: Ansoft, HP-HFSS, Raphael**

Integral equation formulation

- **The solution of the inhomogeneous Poisson equation:**

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon}$$

written in terms of the Green's function:

$$\Phi(x) = \int_{R'} G(x, x')\rho(x)(x')dR'$$

- **Analogous to the circuit impulse response, satisfies:**

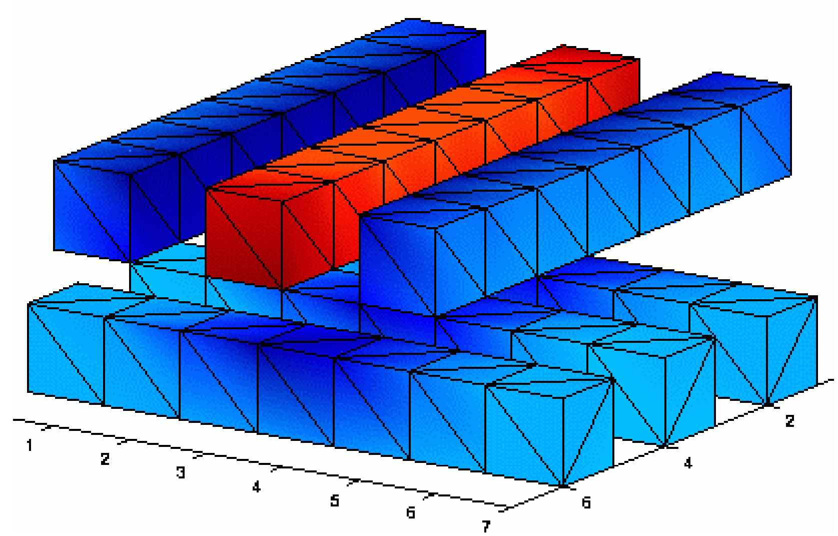
$$\varepsilon \nabla^2 G(x, x\ ) = -\delta(x - x\ )$$

sometimes solvable analytically, e.g., in free space:

$$G(x, x') = \frac{1}{4\pi\varepsilon \|x - x'\|}$$

Discretization of the integral equation

- Charge accumulates only on conductor surfaces => Integration performed over conductor surfaces



$$\Phi(x) = \int_{\Sigma} G(x, x')\rho_s(x')ds$$

- Discretize using e.g. first-order collocation scheme (better methods exist)
- The integral equation is reduced to $\mathbf{A}\sigma = \varphi$

where $$\mathbf{A} = \{a_{ij}\} = \int_{\Sigma_j} G(x_i, x')ds \qquad 1 \le i, j \le M$$

$$\varphi = \{\varphi_i\} = \sum_{j=1}^{M} \rho_s(x_j) \int_{\Sigma_j} G(x_i, x')ds \qquad 1 \le i \le M$$

Integral Equations

1. **Surface Meshing of structures**
2. **Smaller but dense matrices**
3. **Matrices are well conditioned**
4. **For exterior problems, easy to enforce boundary conditions**
5. **Computationally expensive**

- **Used for planar layered media (e.g., CMOS processes)**
- **Commercial tools: Sonnet, Momentum, Strata**

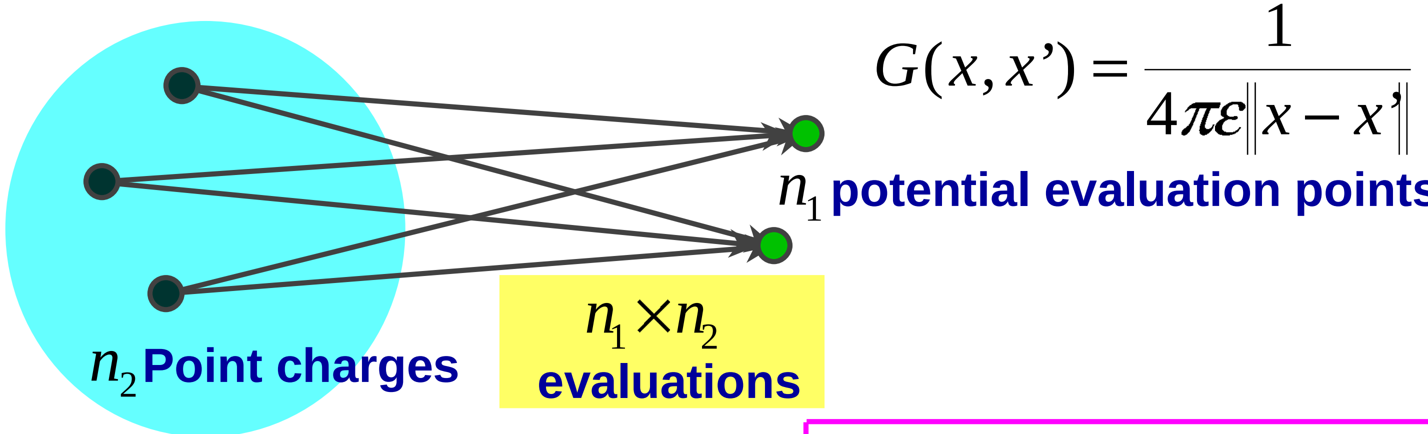
Standard dense matrix solutions

- **Gaussian Elimination**
  - —Direct LU factorization of the matrix
  - —Memory requirements: $O(N^2)$ storage
  - —CPU requirements: $O(N^3)$ operations
  - —Prohibitive. Handles only small problems $(N < 2000)$
- **Krylov Iterative Methods (e.g., CG, GMRES, QMR)**
  - —Involves a sequence of matrix vector multiplications
  - —Memory requirements: $O(N^2)$ storage
  - —CPU requirements: $O(k N^2)$
  - —Applicable to medium problems $(N < 4000)$. Large problems intractable because of memory requirements

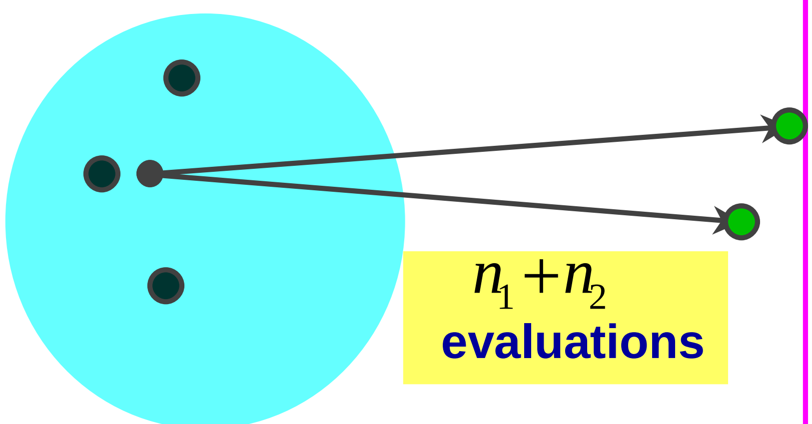
Multipole accelerated matrix solution

- **FastCap (MIT), specific to free-space Green function kernel**

$$G(x, x') = \frac{1}{4\pi\varepsilon \|x - x'\|}$$



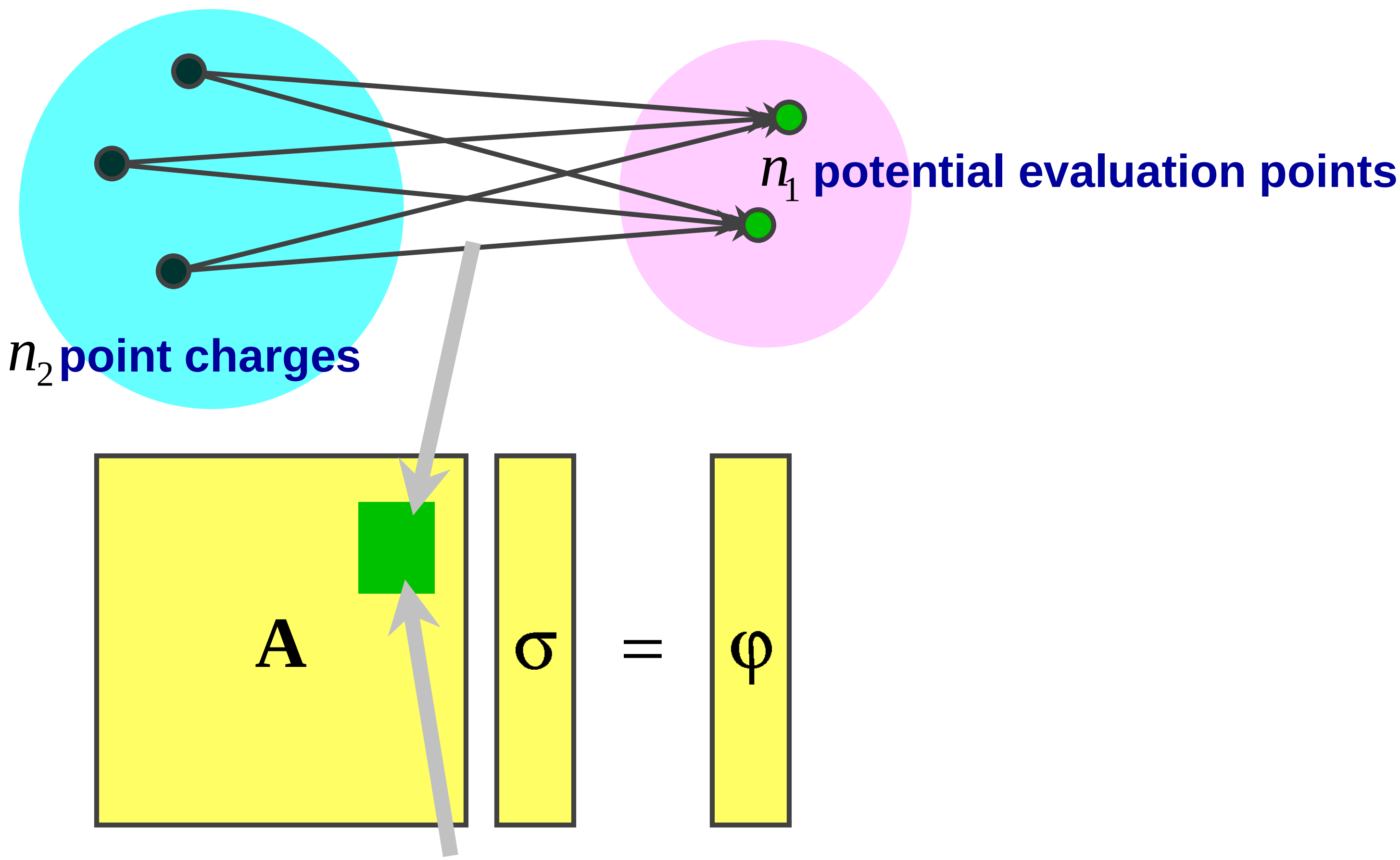
- **monopole approximation**



- **Monopole is first term of multipole expansion => with more terms one can compute potentials to required accuracy**
- **Distant interactions computed with few expansion terms**
- **Matrix-vector product O(N) time and storage**

SVD accelerated matrix solution

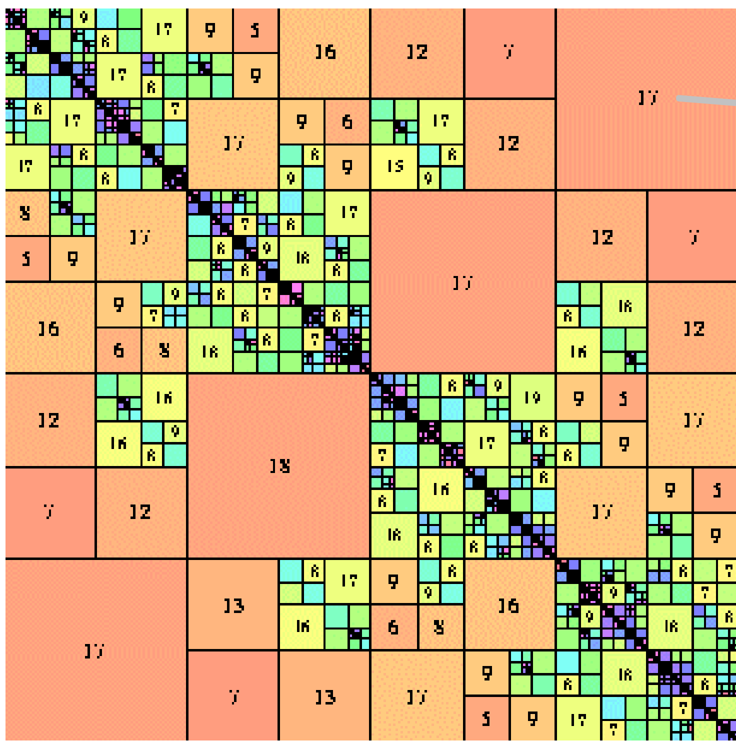
- **IES^3, kernel independent, applicable to general kernels, i.e., from layered-media Green's function.**
- **Intuition:**



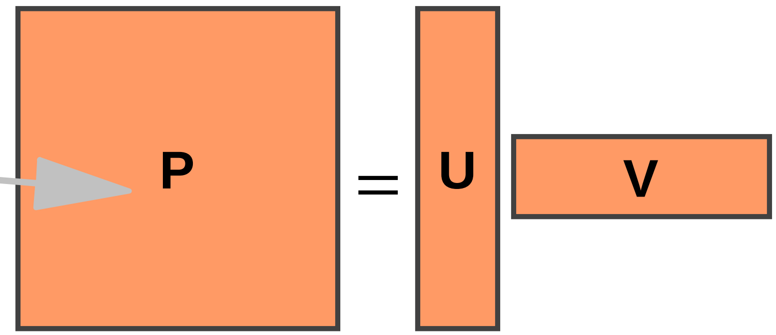
Well-separated points => low rank sub-matrix

SVD matrix compression

Properly reordered A matrix sub-block ranks



- **SVD factorization of *N x N* matrix P of rank *r***



U and V are *N x r* matrices

- **Matrix-vector product**
  - **— Directly: requires *O(N^2)* operations**
  - **— Using the UV representation requires *2 r N* operations**
- ***When r << N this is far more efficient***

Integral equations + matrix acceleration

- **Krylov Iterative Methods (e.g., CG, GMRES, QMR)**
  - **— Involves a sequence of matrix vector multiplications.**
- **Memory requirements: <O(NlogN) storage.**
- **CPU requirements: <O(k NlogN).**
- **Applicable to large problems (N ~ 10^5).**
- **Multipole acceleration specific to free-space kernel.**

$$G(x, x') = \frac{1}{4\pi\varepsilon \|x - x'\|}$$

- **SVD compression more general**
  - **— valid for any "smooth" kernel.**
  - **— suitable for the CMOS multilayer dielectric media.**

The random walk method

Key result (theorem):

- **The Laplace equation**

$$\nabla^2 \Phi = 0$$

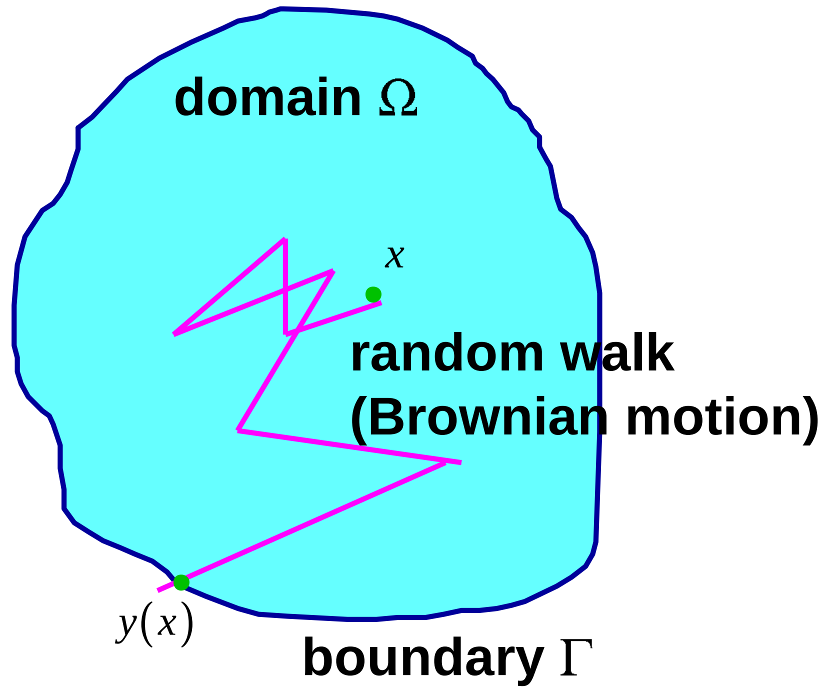
with boundary conditions

$$\Phi|_\Gamma = f$$

$y(x)$ **the first exit point of a random walk starting in** x

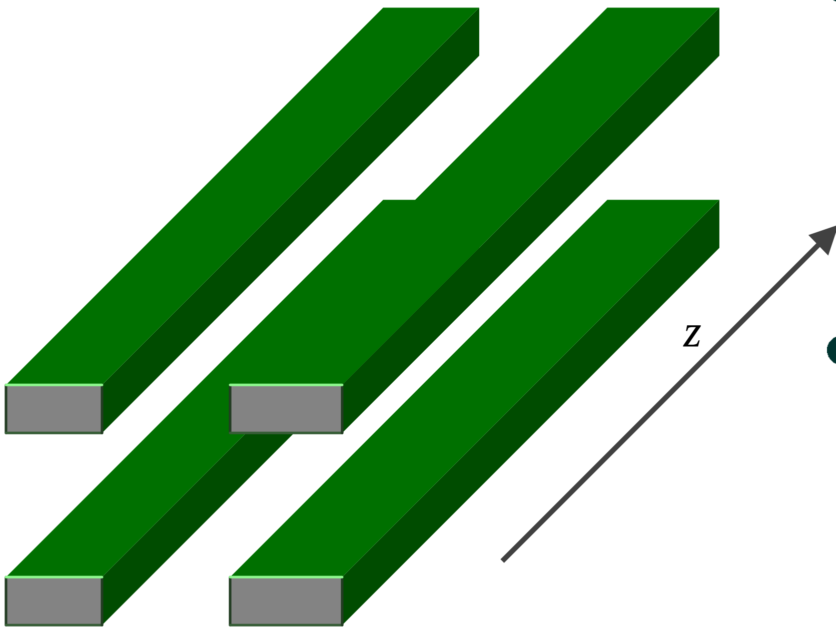
has the solution:

$$\overline{\Phi}(x) = \mathrm{E}\left[f\left(y(x)\right)\right]$$



- **Implemented in QuickCap**
- **Advantageous**
  - — for high-dimensional domains with complex boundaries
  - — when solution is sought only in some points of interest

Inductance computation



- high frequency approximation
- perfect conductor (0 skin depth)
- all modes have same propagation velocity v

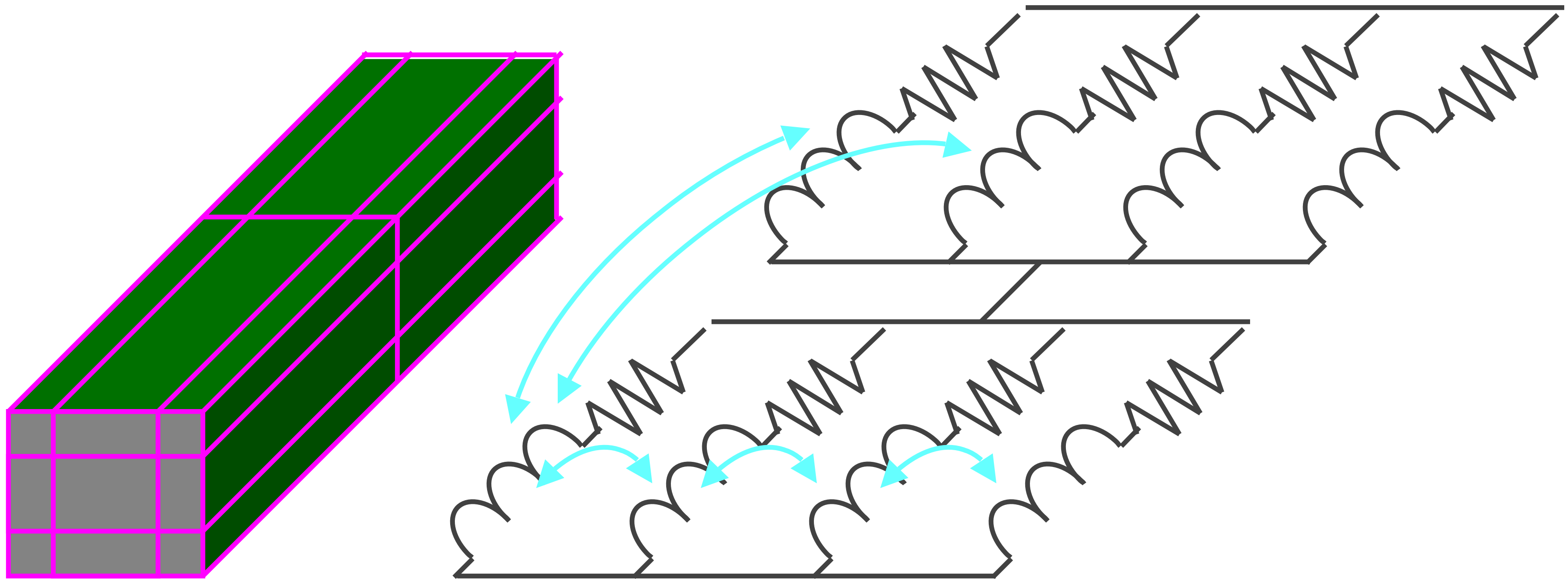
$$[\mathbf{L}] = \frac{1}{v^2}[\mathbf{C}]^{-1}$$

- in layered media use:

$$[\mathbf{L}] = \frac{1}{c^2}[\mathbf{C}_0]^{-1}$$

- Inductance computed using capacitance computation methods

PEEC discretization for inductance



- Dense inductance matrix:

$$M_{ij} = \frac{\mu}{4\pi} \iint \frac{dl_i \cdot dl_j}{R}$$

- Multipole (FastHenry) or SVD techniques must be used,
- Captures skin effect if discretization is fine enough.