PROBABILITY-ONE HOMOTOPY METHODS FOR NONSMOOTH NONLINEAR SYSTEMS OF EQUATIONS

Layne T. Watson
Departments of Computer Science and Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0106 USA

http://www.cs.vt.edu/~ltw/

Virginia Tech

OUTLINE

1. Background—a homotopy methods tutorial.
2. Software—HOMPACK90, POLSYS_PLP.
3. Application to circuit design.
4. Nonsmooth functions—definitions.
5. Probability-one homotopy theory for nonsmooth functions.
6. Application—mixed complementarity problem.

BACKGROUND

Problem: given C^2 $F : E^n \to E^n$, solve $F(x) = 0$.

Definition. Let $U \subset E^m$ and $V \subset E^n$ be open, and $\rho : U \times [0,1) \times V \to E^n$ be C^2. ρ is said to be *transversal to zero* if $D\rho$ has full rank on $\rho^{-1}(0)$.

Parametrized Sard's Theorem. Let $\rho : U \times [0,1) \times V \to E^n$ be C^2 and define

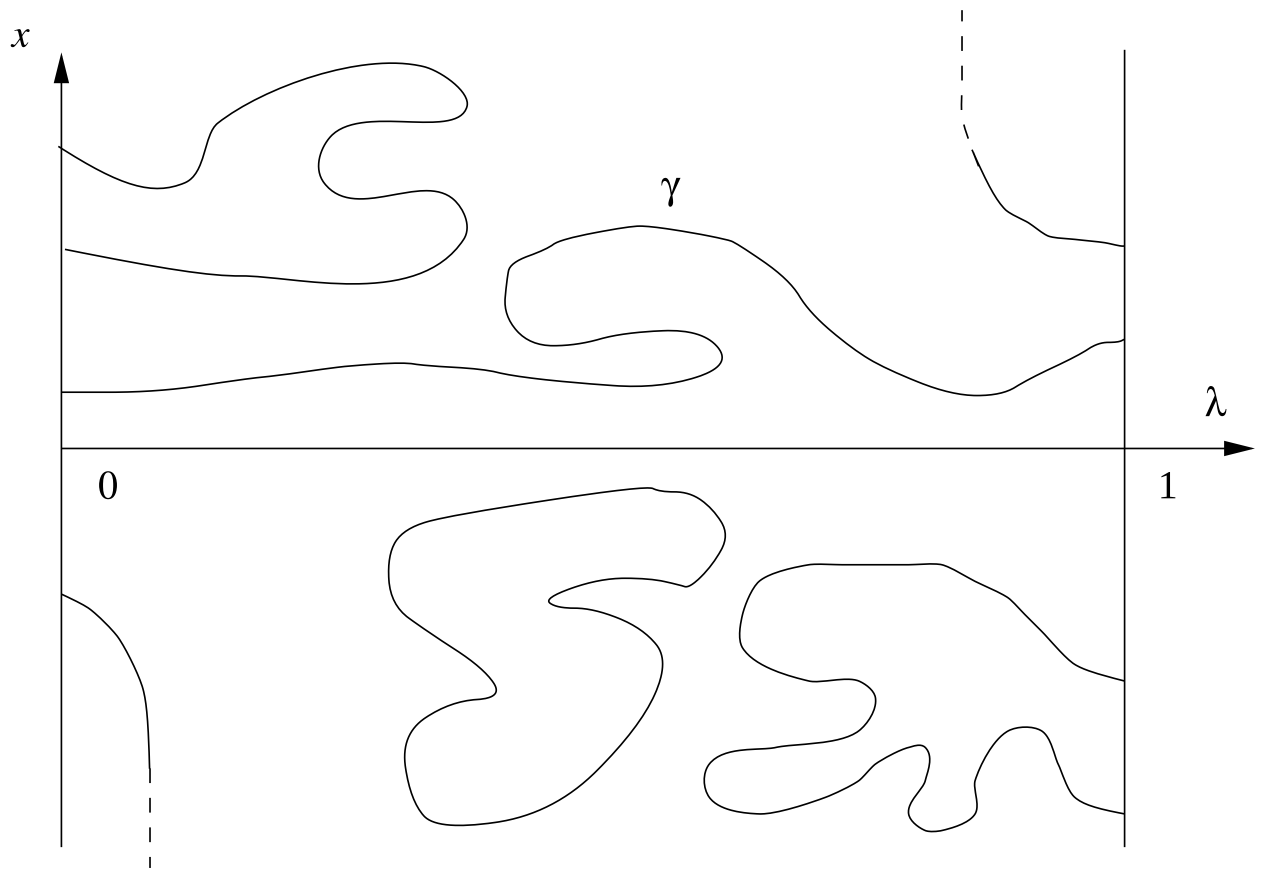
$$\rho_a(\lambda, x) = \rho(a, \lambda, x).$$

If ρ is transversal to zero, then for almost all $a \in U$, ρ_a is also transversal to zero.

Typical choices for ρ_a:

$$\rho_a(\lambda, x) = \lambda\, F(x) + (1 - \lambda)(x - a),$$
$$\rho_a(\lambda, x) = \lambda\, F(x) + (1 - \lambda)\, G(x; a),$$
$$\rho_a(\lambda, x) = F(x) - (1 - \lambda)\, F(a).$$

BACKGROUND



Typical $\rho_a^{-1}(0)$ for $\rho_a(\lambda, x)$ transversal to zero.

GLOBALLY CONVERGENT PROBABILITY-ONE HOMOTOPY ALGORITHM

Given open $U \subset E^m$, open $V \subset E^n$, and C^2 $F : V \to E^n$, choose $\rho : U \times [0,1) \times V \to E^n$ such that

(1) $\rho(a, \lambda, x)$ is C^2 and transversal to zero, and

for each $a \in U$,

(2) $\rho_a(0, x) = 0$ has a unique solution x^a at which rank $D_x \rho_a(0, x^a) = n$,

(3) $\rho_a(1, x) = F(x)$,

(4) $\rho_a^{-1}(0)$ is bounded.

Then for almost all $a \in U$ there exists a zero curve γ of ρ_a, along which $D\rho_a$ has full rank, emanating from $(0, x^a)$ and reaching a point $(1, \bar{x})$ where $F(\bar{x}) = 0$. γ has finite arc length if $DF(\bar{x})$ is invertible. γ does not intersect itself, and is disjoint from all other zeros of ρ_a.

SOFTWARE — HOMPACK90

- Fortran 90 modules.
- Specialized code for $x = f(x)$, $F(x) = 0$, and $\rho(a, \lambda, x) = 0$.
- Three different curve tracking algorithms (ODE based, normal flow, augmented Jacobian matrix).
- Special algorithms for small dense and large sparse Jacobian matrices.
- Easy to use code for polynomial systems.

SOFTWARE — POLSYS_PLP

- For polynomial systems with complex coefficients.
- Implements state-of-the-art theory for exploiting structure.
- Elegant interface for defining system and its structure.
- Fortran 90 modules.

CIRCUIT DESIGN APPLICATION

Variable stimulus homotopy:

$$\rho_a(\lambda, x) = \lambda \, F(x, \lambda) + (1 - \lambda) G(x, a).$$

$F(x, \lambda)$ turns on nonlinear circuit components as λ increases from 0 to 1. $G(x, a)$ models a simple linear circuit. $\rho_a(\lambda, x)$ models **some** no gain circuit for all $0 \le \lambda \le 1$.

Reference: R. C. Melville, Lj. Trajković, S.-C. Fang, and L. T. Watson, “Artificial parameter homotopy methods for the DC operating point problem”, *IEEE Trans. Computer-Aided Design*, 12 (1993) 861–877.

NONSMOOTH FUNCTIONS

Let $F : E^n \to E^n$ be locally Lipschitzian (hence differentiable on a dense set D_F). The B-subdifferential is

$$\partial_B F(x) := \left\{ V \;\middle|\; \exists \{x^k\} \to x,\ x^k \in D_F,\ \text{with } V = \lim_{k\to\infty} \nabla F(x_k) \right\}.$$

The Clarke subdifferential $\partial F(x)$ is the convex hull of $\partial_B F(x)$. F is said to be *semismooth* at x if it is directionally differentiable at x and for any $V \in \partial F(x+h)$, $h \to 0$,

$$Vh - F'(x;h) = o(\|h\|).$$

F is said to be *strongly semismooth* if additionally,

$$Vh - F'(x;h) = \mathcal{O}(\|h\|^2).$$

A semismooth function $F : E^n \to E^n$ is *BD-regular* at x if all elements in $\partial_B F(x)$ are nonsingular, and F is *strongly regular* at x if all elements in $\partial F(x)$ are nonsingular.

HOMOTOPY THEORY FOR NONSMOOTH FUNCTIONS

Theorem. Let $F : E^n \to E^n$ be a Lipschitz continuous function and suppose there is a C^2 map

$$\rho : E^m \times [0,1) \times E^n \to E^n$$

such that

1. $\nabla\rho(a,\lambda,x)$ has rank n on the set $\rho^{-1}(\{0\})$,
2. the equation $\rho_a(0,x) = 0$, where $\rho_a(\lambda,x) := \rho(a,\lambda,x)$, has a unique solution $x^a \in E^n$ for every fixed $a \in E^m$,
3. $\nabla_x\rho_a(0,x^a)$ has rank n for every $a \in E^m$,
4. ρ is continuously extendible to the domain $E^m \times [0,1] \times E^n$, and $\rho_a(1,x) = F(x)$ for all $x \in E^n$ and $a \in E^m$, and
5. γ_a, the connected component of $\rho_a^{-1}(\{0\})$ containing $(0,x^a)$, is bounded for almost all $a \in E^m$.

Then for almost all $a \in E^m$ there is a zero curve γ_a of ρ_a, along which $\nabla\rho_a$ has rank n, emanating from $(0,x^a)$ and reaching a zero $\bar{x}$ of F at $\lambda = 1$. Further, γ_a does not intersect itself and is disjoint from any other zeros of ρ_a. Also, if γ_a reaches a point $(1,\bar{x})$ and F is strongly regular at $\bar{x}$, then γ_a has finite arc length.

SMOOTHING OPERATORS

Definition. Given a nonsmooth continuous function $\phi : E^p \to E$, a *smoother* for ϕ is a continuous function $\tilde{\phi} : E^p \times [0, \infty) \to E$ such that

1. $\tilde{\phi}(x, 0) = \phi(x)$, and
2. $\tilde{\phi}$ is continuously differentiable on the set $E^p \times (0, \infty)$.

If $\tilde{\phi}$ is C^2 on $E^p \times (0, \infty)$, call $\tilde{\phi}$ a C^2-*smoother*. For brevity, write $\phi_\mu(x) := \tilde{\phi}(x, \mu)$.

Example: $\tilde{\phi}(a, b, \mu) := a + b - \sqrt{a^2 + b^2 + 2\mu}$ is a smoother for the NCP (nonlinear complementarity problem) function $\phi(a, b) := a + b - \sqrt{a^2 + b^2}$, where

$$\phi(a, b) = 0 \iff 0 \leq a \perp b \geq 0.$$

APPLICATION TO COMPLEMENTARITY PROBLEMS

Nonlinear complementarity problem *NCP(G)*: find $x \in E^n$ such that $x \geq 0$, $G(x) \geq 0$, $x^t G(x) = 0$.

Define $F : E^n \to E^n$ by $F_i(x) = \phi(x_i, G_i(x))$, $F^\mu : E^n \to E^n$ by $F_i^\mu(x) = \phi_\mu(x_i, G_i(x))$. Then F^μ is a smoother for F, and x solves *NCP(G)* if and only if

$$F(x) = 0.$$

Let $\mu : [0,1] \to [0,\infty)$ be a C^2 decreasing function with $\mu(1) = 0$. Solve *NCP(G)* by using the homotopy map

$$\rho_a(\lambda, x) = \lambda\, F^{\mu(\lambda)}(x) + (1-\lambda)(x-a).$$

REFERENCES

S. C. Billups and L. T. Watson, "A probability-one homotopy algorithm for nonsmooth equations and mixed complementarity problems", *SIAM J. Optim.*, 12 (2002) 606–626.

R. C. Melville, Lj. Trajković, S.-C. Fang, and L. T. Watson, "Artificial parameter homotopy methods for the DC operating point problem", *IEEE Trans. Computer-Aided Design*, 12 (1993) 861–877.

L. T. Watson, M. Sosonkina, R. C. Melville, A. P. Morgan, and H. F. Walker, "Algorithm 777: HOMPACK90: A suite of FORTRAN 90 codes for globally convergent homotopy algorithms", *ACM Trans. Math. Software*, 23 (1997) 514–549.

L. T. Watson, "Probability-one homotopies in computational science", Technical Report TR 00-03, Dept. of Computer Sci., Virginia Polytechnic Institute and State Univ., Blacksburg, VA, 2000. http://www.cs.vt.edu

S. M. Wise, A. J. Sommese, and L. T. Watson, "Algorithm 801: POLSYS_PLP: A partitioned linear product homotopy code for solving polynomial systems of equations", *ACM Trans. Math. Software*, 26 (2000) 176–200.