Network Tomography via Compressed Sensing

Mohammad H. Firooz, *Student Member, IEEE,* Sumit Roy, *Fellow, IEEE,*
Electrical Engineering Department
University of Washington, Seattle, WA, USA.
{firooz,sroy}@u.washington.edu

***Abstract*—In network tomography, we seek to infer link status parameters (such as delay) inside a network through end-to-end probe sending between (external) boundary nodes. The main challenge here is to estimate link-level attributes from end-to-end measurements. In this paper by using the idea of combinatorial compressed sensing, we provide conditions on network routing matrix under which it is possible to estimate links delay from end-to-end delay measurements. We also provide an upper-bound on the estimation error. Moreover, for a given network, we show how to design its routing matrix to achieve the minimum number of probes needed to be sent in order to estimate delay of the links inside the network.**

I. Introduction

Monitoring of link properties (delay, loss rates etc.) within the Internet has been stimulated by the demand for network management tasks such as fault and congestion detection or traffic management. This would help network engineers and Internet Service Providers (ISP) to keep track of network utilization and performance. The need for accurate and fast network monitoring method has increased further in recent years due to the complexity of new services (such as video-conferencing, Internet telephony, and on-line games) that require high-level quality-of-service (QoS) guarantees. In 1996, the term *network tomography* was coined by Vardi [1] to encompass these class of approaches that seek to infer internal link parameters and identify link congestion status.

Current network tomography methods can be broadly categorized as follows:

- **Node-oriented:** These methods are based on cooperation among network nodes on an end-to-end route using *control* packets. For example, active probing tools such as ping or traceroute, measure and report attributes of the round-trip path (from sender to receiver and back) based on separate probe packets[2]. The challenges of such node-oriented methods arise from the fact that many service providers do not own the entire network and hence do not have access to the internal nodes[3].
- **Edge-oriented:** In networks with a defined *boundary*, it is assumed that access is available to all nodes at the edge (and not to any in the interior). A boundary node sends probes to all (or a subset) of other boundary nodes to measure packet attributes on the path between network end-to-end points. Clearly, these *edge-based* methods do not require exchanging special control messages between interior nodes. The primary challenge of such end-to-end probe data [4],[5] to estimate *link level* attribute is that of *identifiability*, as will be discussed later.

As the Internet evolves towards decentralized, uncooperative, heterogeneous administration and edge-based control, node-oriented tools will be limited in their capability. Accordingly, in this work we only focus on edge-oriented methods which have recently attained more attention due to their ability to deal with uncooperative and heterogeneous (sub)networks.

In edge-oriented network tomography, probes are sent between two boundary nodes on pre-determined routes; typically these are the shortest paths between the nodes. For some parameters like delay (which are the main concern in this manuscript), an additive linear model adequately captures the relation between end-to-end and individual link delays, and can be written as [6], [7]

$$\mathbf{y} = \mathbf{R}\mathbf{x} \tag{1}$$

where $\mathbf{x}$ is the $n\times 1$ vector of individual link delays. The $r \times n$ *binary* matrix $\mathbf{R}$ denotes the routing matrix for the network graph corresponding to the measurements and $\mathbf{y} \in \mathbb{R}^r$ is the measured r-vector of end-to-end path delays. Solution approaches based on Eq. (1) can be largely categorized as follows:

1) **Deterministic models**: Here the link attributes, such as link delay, are considered as unknown but constant; the goal of network tomography is to estimate the value of those constants. Since the link delay is typically time varying in any network, this approach is suitable for periods of local 'stationarity' where such an assumption is valid.
2) **Stochastic model**: Here, it is supposed that the link vector $\mathbf{x}$ is specified by a suitable probability distribution. The goal of network tomography is to identify the unknown parameters of the probability model. For example, many works assume the link attributes follow a Gaussian distribution or an exponential distribution [8], [6], [7]. Further, the observations are assumed to occur in the presence of an independent additive noise or interference term ϵ [9]; thus the observation equation is modified to $\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon$.

There exist challenges with both modeling approaches. Our work falls within the class of deterministic approaches. Stochastic approaches in the literature are Bayesian in nature, requiring a prior distribution. If incorrectly chosen, this lead to biases in the resulting estimates. Further, stochastic models are usually more computationally intensive than deterministic ones [10]. On the other hand, deterministic models suffer from generic *identifiability* problems; this will be discussed subsequently in more detail.

In Eq. (1), typically, the number of observations $r \ll n$, because the number of accessible boundary nodes is much smaller than number of links inside the network. Thus the number of variables in Eq. (1) to be estimated is much larger than number of equations in the linear model ($rank(\mathbf{R}) < n$)[9], leading to generic non-uniqueness for any solution to Eq. (1),i.e., inability to uniquely specify links delay [8].

A potential solution to the above problem is to limit ourselves to links with large delays and try to *best* estimate their values. That is because network administrators are mainly interested in locating deficient links or links with high traffic inside the network. These sorts of links have significantly high delays (or high packet lost rate) comparing to the other links having negligible delay (or low packet lost rates). Another reasonable assumption to make in order to solve the underdetermined system in Eq. (1) is that number of links with large delays (or high path lost rates) is relatively small compare to total number of links inside the network. It means vector $\mathbf{x}$ in that equation has a few large entries, up to k, which we are interested to estimate. We refer to such a vector a *k-compressible* vector.

In this manuscript, by using the concept of expander graphs and binary compressed sensing, which is a new research avenue, and k-compressible assumption, we provide conditions on routing matrix of a network which gives the "*best*" estimation of links delay from end-to-end delay measurements. The "*best*" estimation here means that the difference between actual links delay and the solution to Eq. (1) goes to zero when the delay of ordinary links (links with negligible delay) tends towards zero. In this case, we call the network k-identifiable (an official definition of k-identifiability will be given later in *Definition* 1). In addition, we shall show that if network is k-identifiable, the underdetermined system of equations in Eq. (1) can be solved using a LP optimizer.

End-to-end delay measurements using probe transmission compels extra burden on the network which as a result affects links delay and lost rate inside the network. This phenomena not only affects the estimation accuracy but also decreases network performance. For that reason, one should minimize total number of probes used for network monitoring. One way of decreasing number of probes injected to the network is to minimize number of paths over which the monitoring probes traverse. For the first time (up to authors knowledge), we model this problem as a binary integer programming problem. Although binary integer programming is an NP-hard problem, it is a well-studied area and there are a number of heuristic algorithms in the literature which have a good approximation results to the main problem.

Our specific cintributions in this work are summerized next:

- We establish a connection between network tomography and binary compressed sensing using expander graphs which has received significant interest during the past few years.
- We provides conditions on routing matrix of a network under which the network is 1-identifiable. Moreover, we provide an upper-bound on estimation error in links delay when network in 1-identifiable.
- Based on our result for 1-identifiability, we show how to design routing matrix for a given network such that the network is 1-identifiable with minimum number of end-to-end transmitted probes.

A. Model and definition

A communication network consisting of bidirectional links connecting transmitters, switches, and receivers can be modeled as an undirected graph $N(V, E)$ where V is a set of vertices and E is a set of edges. Let $B \subset V$ be a set of boundary nodes which we have access to. From these boundary nodes a set of measurements is taken by using end-to-end probe sending methods which is represented by $\mathbf{y}$ in this paper. As mentioned before, there is a linear relation between the measurements $\mathbf{y}$ and delays of the links given in Eq. (1).

As discussed before we assume that vector $\mathbf{x}$ in Eq. (1) is k-compressible; i.e. it has up to k large values while the others are close enough to zero (relatively). Having this assumption we define a network $N(V, E)$ to be k-identifiable as below:

Definition 1. *Network $N(V, E)$ with routing matrix $\mathbf{R}$ is called k-identifiable (under end-to-end probe sending method) if for any delay vector $\mathbf{x}^*$ which is k-compressible the following holds:*

$$\| \mathbf{x} - \mathbf{x}^* \|_1 \rightarrow 0 \text{ as } \| \mathbf{x}^* - \mathbf{x}^*_S \|_1 \rightarrow 0 \qquad (2)$$

where $\mathbf{x}^*$ is actual links delay inside the network, $\mathbf{x}$ is a solution to Eq. (1) *and $\mathbf{x}^*_S$ is k large values of vector $\mathbf{x}^*$.*

In Eq. (2), $\| \mathbf{x}^* - \mathbf{x}^*_S \|_1 \rightarrow 0$ means that delay of ordinary links (links with negligible delay) goes to zero. Note that if $\| \mathbf{x}^* - \mathbf{x}^*_S \|_1 = 0$ it means delay vector $\mathbf{x}^*$ contains only up to k nonzero entries, or in other words $\mathbf{x}^*$ is k-sparse. Almost all works in the literature is based on this assumption. In other words, They call a network k-identifiable if Eq. (1) is uniquely solvable given that there are only up to k nonzero elements in $\mathbf{x}$. Clearly, our definition of identifiability is an extension to the old definition and it is more realistic since delay of other links are close to zero but not exactly zero.

In this paper for the sake of simplicity we only consider the case of $k = 1$. This is the simplest possible class of identifiability problems (compared to the general and more difficult $k > 1$ case) and yet is sufficiently challenging as our investigations will show.

The paper is organized as follows: Section II relates links delay estimation to binary compressed sensing and gives condition on network routing matrix under which a given network is 1-identifiable. In section III, we look at minimizing number of probes sent to the network subject to 1-identifiability of the network. The paper concludes with reflections on future work in Section IV.

II. Expander Graph and Network Identifiability

In this section we establish a connection between identifiability in a network $N(V, E)$ to the recently developed concept of compressed sensing using expander graphs. This connection makes it possible to use results in this new field , attracted lots of attention these days, to network tomography.

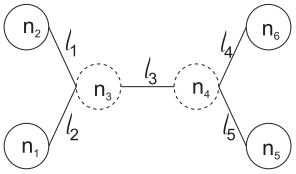

Fig. 1. A network with 4 boundary nodes, 2 intermediate nodes and 5 links

In Figure 1 a toy network with 4 boundary nodes and 2 intermediate node is depicted. Throughout this manuscript, boundary nodes are depicted as solid circles while intermediate nodes are presented using dashed circle. Network in Figure 1 is used to illustrate definitions and theorems in the following sections.

A. Routing Matrix and Bipartite Graph

A bipartite graph is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y; A bipartite graph is usually presented as a triple $G(X, Y, H)$ where $H \subset X \times Y$ is a set of edges between two parts. Sets X and Y are called left side and right side of the graph, respectively. A bipartite graph $G(X, Y, H)$ can also be represented with a matrix $T = [t_{ij}]$, known as bi-adjacency matrix, where $t_{ij} = 1$ if node i in X is connected to node j in Y, or equivalently if $(i, j) \in H$ and it is zero otherwise.

Suppose network $N(V, E)$ is given. Let n be the number of links in this network ($n = |E|$), $\mathcal{R}$ be the given collection of paths between boundary nodes and r be the cardinality of $\mathcal{R}$, total number of paths between boundary nodes. Further, let $\mathbf{R}_{r\times n}$ be the given routing matrix which gives the accessible path from one boundary node to another. Path collection $\mathcal{R}$ and routing matrix $\mathbf{R}$ are equivalent, i.e. both have the same information about existing paths between boundary nodes inside the network.

For our example in Figure 1, suppose the following routing matrix is given:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{3}$$

which is equivalent to paths collection $\mathcal{R}$ as follows:

$$\mathcal{R} = \{l_1 l_3 l_5, l_2 l_3 l_4, l_1 l_2, l_4 l_5\} \tag{4}$$

Note that the above routing matrix, or its equivalent paths collection, is not a complete routing matrix of network in Figure 1. For instant it doesn't include the path from n_1 to n_6 which is $l_2 l_3 l_4$. However, it is a fundamental assumption in network tomography that the routing matrix is already given, as part of the problem, and the goal is to use this given routing matrix to estimate links parameters (delay is our case) inside the network. In the next section we talk about cases where we actually can design the routing matrix.

$\mathbf{R}_{r\times n}$ can be thought of a bi-adjacency matrix of a bipartite graph $G(X, Y, H)$ where $X = E$, set of links in network $N(V, E)$, $Y = \mathcal{R}$,set of given paths in the network, and there is a connection between a node in X to a node in Y if the corresponding path in Y goes through the corresponding link in X. Figure 2 presents the corresponding bipartite graph of network given in Figure 1 with routing matrix $\mathbf{R}$ in Eq. (3).

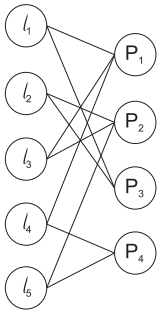

Fig. 2. bipartite graph corresponding to given routing matrix in Eq. (3)

B. Expander Graphs

Over recent years a new approach for obtaining a succinct approximate representation of n-dimensional vectors (or signals) has been discovered. For any signal $\mathbf{x}$, the representation is equal to $\mathbf{Rx}$, where $\mathbf{R}$ is carefully chosen $r \times n$ matrix which is often referred to as *measurement matrix*. The main challenge in compressed sensing area is to design $\mathbf{R}$ with desirable properties, such as maximum possible compression or fast decoding time.

However the problem we are dealing with is the converse of the problem acknowledged in the literature so far. We already have the measurement matrix, which is the routing matrix of the network, and the question is on the competency of that routing matrix for compressed sensing? up to knowledge of the authors there is no study in the literature regarding this issue.

The measurement matrix in a tomography problem is a binary matrix and as it is mentioned before it can be considered as a bi-adjacency matrix of a bipartite graph. Berinde and Indyk in [11] show that bi-adjacency matrix of special bipartite graphs, called expander graphs, can be used as measurement matrix.

Definition 2. *A $(s, d, \epsilon)-expander$ is a bipartite simple graph $G(X, Y, H)$ with left degree d (i.e. $deg(v) = d \forall v \in X$) if for any $S \subset X$ with $|S| \leq s$ the following condition holds:*

$$|N(S)| \geq (1 - \epsilon)d|S| \tag{5}$$

where $N(S)$ is set of neighbors of S.

Roughly speaking, in an expander graph any collection of nodes in the left hand side (X) expands to a sufficiently large number of nodes in the right hand side (Y). Expander graphs are well-studied topic in computer science and mathematics. There are a number of papers in the literature on how to construct a $(s, d, \epsilon) - expander$ graph. Interested readers can refer to [12], [13], [14].

Berinde and Indyk in [15], [11] show that bi-adjacency matrix of a $(2s, d, \epsilon) - expander$ graph can be used as measuring matrix for a s-sparse signal. Therefore, to show a

given network $N(V,E)$ with routing matrix $\mathbf{R}$ is 1-identifiable it is enough to show that the bipartite graph with bi-adjacency matrix $\mathbf{R}$ is a 2-expander graph.

Parameter ϵ in definition of an expander graph is a design parameter which is related to recovering error after compression. Berinde and Indyk derive their results for $\epsilon < \frac{1}{16}$. However, since we use expander graphs as a test tool, to examine whether a given routing matrix is a binary measurement matrix, we need ϵ to be as large as possible to increase the chance of finding identifiable networks. In other words, raising ϵ would expand space of expander graphs which means more bipartite graphs, or as a result more routing matrices, would be considered as expander graphs. On the other hand, as mentioned earlier, increasing ϵ increases error recovery and consequently enhances ambiguity level.

Motivated by above argument, there is a tradeoff between number of networks which are 1-identifiable and also their routing matrix is bi-adjacency of an expander graph with estimation error in links delay. In the following theorems we show that, with reasonable recovering error, ϵ can be increase to $1/4$. As we shall show later the bipartite graph in Figure 2 is an expander graph when $\epsilon = 1/4$. Which means our result includes more 1-identifiable networks while it keeps estimation error over an reasonable level.

Theorem 1. *Let $G(V_1, V_2, E)$ be a $(2,\epsilon)$ expander graph with left degree d. Let $\mathbf{A}_{m\times n}$ be its bi-adjacency matrix. Further assume $\mathbf{w}$ is in null space $\mathbf{A}$ ($\mathbf{Aw} = \mathbf{0}$) and let S be any set of $k = 1$ coordinates of $\mathbf{w}$. Then*

$$\| \mathbf{w}_S \|_1 \leq 2\epsilon \| \mathbf{w}_{S^c} \|_1 \tag{6}$$

proof: See [16].

The following theorem puts an upper bound on error of recovering $\mathbf{x}$ from its linear projection $\mathbf{Ax}$ when $\mathbf{A}$ is a bi-adjacency matrix of an $(2,d,\epsilon)$-expander graph.

Theorem 2. *Consider any two vectors $\mathbf{x}$, $\mathbf{x}'$, such that they have the same projection under measurement matrix $\mathbf{A}$; i.e. $\mathbf{Ax} = \mathbf{Ax}'$. Further, suppose $\| \mathbf{x}' \|_1 \leq \| \mathbf{x} \|_1$. Let S be the set of $k = 1$ largest (in magnitude) coefficients of $\mathbf{x}$. Then*

$$\| \mathbf{x}' - \mathbf{x} \|_1 \leq f(\epsilon) \| \mathbf{x}_{S^c} \|_1 \tag{7}$$

where $f(\epsilon) = \frac{1+2\epsilon}{2(1-2\epsilon)}$

The following theorem relates delay estimation in a given network $N(V,E)$ to the results of expander graphs. In addition, it proves that end-to-end measurement equation in (1) can be solved using a LP optimizer.

Theorem 3. *Let $N(V,E)$ be a network with paths collection $\mathcal{P}$ and routing matrix $\mathbf{R}$. Suppose $G(E,\mathcal{P},H)$ is a bipartite graph with biadjacency matrix $\mathbf{R}$. Assume $\mathbf{x}^*$ is delay vector of $N(V,E)$. Further assume $\mathbf{x}$ be a solution to the following LP optimization:*

$$\begin{aligned} &\min \| x \|_1 \\ &\textit{s.t.} \\ &\mathbf{Rx} = \mathbf{Rx}^* \end{aligned} \tag{8}$$

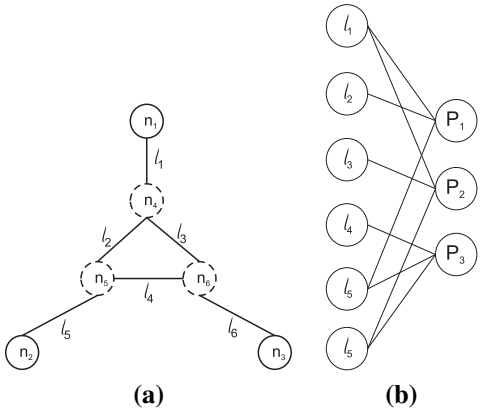

Fig. 3. An example of network which is 1-identifiable but its corresponding bipartite graph is not an expander graph (a) Network topology (b) Its corresponding bipartite graph

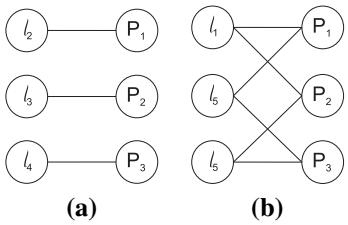

Fig. 4. Two subgraphs of bipartite graph in Figure 3-b which are regular in their left side

Then

$$\| \mathbf{x} - \mathbf{x}^* \|_1 \rightarrow 0 \textit{ as } \| \mathbf{x}^* - \mathbf{x}^*_S \|_1 \rightarrow 0$$

if G is a $(2,d,\epsilon)$-expander where $\epsilon \leq \frac{1}{4}$.

In other words the network, $N(V,E)$, is 1-identifiable if G is a $(2,d,\epsilon)$-expander for $\epsilon \leq \frac{1}{4}$.

proof: See [16].

For example, as mentioned before, The bipartite graph in Figure 2 is proven to be $(2,2,\frac{1}{4}) - expander$ which means network in Figure 1 with routing matrix given in Eq. (3) is 1-identifiable.

It should be mentioned that in general the reverse of Theorem 3 is not true; i.e. there exists some networks $N(V,E)$ which are 1-identifiable but their corresponding bipartite graph is not an expander graph. An example of such a graph is depicted in Figure 3-a. The bipartite graph of its routing matrix is presented in Figure 3-b. As you can see this bipartite graph is not regular in its left side which means it cannot be an expander graph (note that regularity is part of definition of an expander graph). The degree of a node in left set is either one or two. Figure 4-(a) and (b), respectively, present subgraph of G with regular left degree one and two. Each of these subgraphs are expander.

Above observation clears necessity of extending result in Theorem 3 for networks whose corresponding bipartite graph is not regular (and therefore is not expander) but it consists of bunch of expander graphs.

Theorem 4. *Let $N(V,E)$ be a network with routing matrix $\mathbf{R}$. Let $G(X,Y,H)$ be a bipartite graph with bi-adjacency matrix $\mathbf{R}$. Suppose $G_i(X_i,Y,H_i)$, $i = 1,2,...M$ be d_i-regular bipartite subgraphs of G such that:*

- $X = \cup X_i$
- $H = \cup H_i$
- $d_i \neq d_j$ *for* $i \neq j$

Then, $N(V, E)$ *is* 1*-identifiable, if each of* G_i *is an* $(2, d_i, \epsilon)$*-expander graph for* $\epsilon \leq \frac{1}{4}$*. Further, links delay is the solution to LP optimization in Eq.* (9)

proof:see the [16].

Basically above theorem says that a network $N(V, E)$ with routing matrix $\mathbf{R}$ is 1-identifiable if every two links l_i and l_j in N either have different degrees in bipartite graph G, which means they belong to different expander graphs, or they satisfy expansion property in Eq. (5). We state this observation in more formal way in the following corollary for our future reference.

Corollary 5. *Let* $N(V, E)$ *be a network with routing matrix* $\mathbf{R}$ *and paths collection* $\mathcal{P}$*. Let* $G(E, \mathcal{P}, H)$ *be its corresponding bipartite graph with bi-adjacency matrix* $\mathbf{R}$*. Then one and only one of the following statement is true for every two links* l_1 *and* l_2 *in* E*:*

- $deg(l_1) > deg(l_2)$
- $deg(l_1) < deg(l_2)$
- $deg(l_1) + deg(l_2) - 4deg(l_1, l_2) \geq 0$

Up to this point, our goal was to figure out if a given network with a given routing matrix is identifiable or not. But sending probes in network may have some cost for network manager. So besides identifiability problem minimizing the cost of identifiability is an important problem. In the next section we shall talk about minimizing number of paths which are sufficient for network identifiability. Minimizing number of paths used for network identifiability reduces number of probes injected to the network which would decrease the cost.

III. Minimum paths selection

A. Network Covering

In network depicted in Figure 1 there are actually 6 paths between boundary nodes:

$$\mathcal{P} = \{l_1 l_3 l_4, l_1 l_3 l_5, l_2 l_3 l_4, l_2 l_3 l_5, l_1 l_2, l_5 l_6\} \tag{9}$$

Obviously only 4 of them are sufficient to figure out delay of every link inside the network. In general, not all of possible paths between boundary nodes are necessary for network tomography. The goal of this subsection is to show how to select paths from $\mathcal{P}$ such that a desirable goal achieves.

First let make the problem simple. Consider a network $N(V, E)$ with collection of end-to-end paths $\mathcal{P}$. We are looking for the minimum number of paths between boundary nodes such that each link inside the network belongs to at least one of them. In other words, we are looking for minimum number of paths that make a link failure inside the network detectable. Network link failure monitoring is a good application of this problem. Here the goal is to send end-to-end probes to see if there exists a failure link inside the network. Surely, we need to minimize number of probes need to be sent in order to not put burden in the network.

Suppose c_i is the cost corresponding to use path P_i in the network(in the case all the path is equal $c_i = 1 \, \forall i$).

To solve this problem let first define an indicator variable I_{P_i} indicating whether path $P_i \in \mathcal{P}$ is used or not as follows:

$$I_{P_i} = \begin{cases} 1 & P_i \text{ is used} \\ 0 & \text{o.w.} \end{cases} \tag{10}$$

In other words, if we use path P_i in our tomography method I_{P_i} gets the value 1 and otherwise it is zero. Therefore each path is engaged in a *yes-no* or so-called *go-no-go* decision [17].

The problem is minimizing number of paths which is: $\min \sum_{i=1}^{r} I_{P_i}$ over all binary variables I_{P_i} subject to the fact that each link belongs to at least one of paths. Let $\mathbf{I}_P^t = [I_{P_i}]_1^r$ be the vector of path indicators. Then the i-th entry of $\mathbf{R}^t \mathbf{I}_P$ is number of paths go through i-th link. That means each entry of $\mathbf{R}^t \mathbf{I}_P$ should be equal or greater than 1. Therefor our problem is the answer to the following minimization problem:

$$\begin{aligned} &\min \sum_{i=1}^{r} I_{P_i} \\ &\text{s.t.} \\ &\mathbf{R}^t \mathbf{I}_P \geq 1 \\ &I_{P_i} \in \{0, 1\} \end{aligned} \tag{11}$$

The above minimization is called *binary integer programming* which is a well-studied area in mathematics. Although this problem is NP-hard, there are a number of algorithms which approximate the problem very well specially when the constraints are of a network nature[18], [17].

B. Minimum paths for 1-identifiability

Now let look at the identifiability problem. The question is: among all available paths in the network, $\mathcal{P}$, what is the subset of $\mathcal{P}$ with minimum cardinality which guarantees 1-identifiability of the network. Clearly, the goal is to minimize $\sum_{i=1}^{r} I_{P_i}$ over set of paths $\mathcal{P}$ in $N(V, E)$ subject to the fact that network remains identifiable. To keep network identifiable, first condition is that our minimum set should cover the network which means, as discussed before, $\mathbf{R}^t \mathbf{I}_p \geq 1$. For identifiability we use the result of Corollary 5 specifying conditions for each two links $l_i, l_j \in E$ under which the network is guaranteed to be identifiable. Using our definition of path indicator in Eq. (10) those conditions can be rewritten as below:

$$\begin{aligned} \sum_{P_k : l_i \in P_k} I_{P_k} - \sum_{P_k : l_j \in P_k} I_{P_k} &\geq 1 \\ \sum_{P_k : l_j \in P_k} I_{P_k} - \sum_{P_k : l_i \in P_k} I_{P_k} &\geq 1 \\ \sum_{P_k : l_i \in P_k \text{or} l_j \in P_k} I_{P_k} - 4 \sum_{P_k : l_i \in P_k, l_j \in P_k} I_{P_k} &\geq 0 \end{aligned} \tag{12}$$

As mentioned before, for each two links l_i and l_j one and only one of the above inequalities must be satisfied. To write this statement mathematically we use *alternative constraints* which a well-known trick in linear binary programming [17].

For that we introduce three binary variables y_1, y_2, y_3 with the following interpretation:

$$y_i = \begin{cases} 1 & \text{if the } i\text{-th constraint is satisfied} \\ 0 & otherwise \end{cases} \quad (13)$$

Then we rewrite the constraints in Eq. (13) in the following format:

$$n(1-y_1) + \sum_{P_k : l_i \in P_k} I_{P_k} - \sum_{P_k : l_j \in P_k} I_{P_k} \geq 1 \quad (14)$$

$$n(1-y_2) + \sum_{P_k : l_j \in P_k} I_{P_k} - \sum_{P_k : l_i \in P_k} I_{P_k} \geq 1$$

$$n(1-y_3) + \sum_{P_k : l_i \in P_k \text{or} l_j \in P_k} I_{P_k} - 4 \sum_{P_k : l_i \in P_k, l_j \in P_k} I_{P_k} \geq 0$$

$$y_1 + y_2 + y_3 = 1$$

The last equality guarantees that one and only one of y_i's is one and the others are zero. Note that if $y_i = 0$ the i-th constraint become a trivial inequality. Therefore constraints in Eq. (15) is equivalent to the statement that one and only one of constraints in Eq. (13) holds.

Above argument results in the fact that if a subset of $\mathcal{P}$ satisfies equality and inequalities given in Eq. (15) then the network is 1-identifiable using those paths.

The following theorem summarized all of above findings for minimum number of paths which quarantines 1-identifiability of a given network $N(V, E)$.

Theorem 6. *Suppose network $N(V, E)$ with routing matrix $\mathbf{R}$ and path collection $\mathcal{P}$ is given. Then the following optimization finds minimum number of paths which guarantees 1-identifiability of the network.*

$$\min \sum_{i=1}^{r} I_{P_i} \quad (15)$$

s.t.

$$\mathbf{R}^t \mathbf{I}_P \geq 0$$

$$\forall l_i, l_j \in E (l_i \neq l_j)$$

$$n(1-y_{1ij}) + \sum_{P_k : l_i \in P_k} I_{P_k} - \sum_{P_k : l_j \in P_k} I_{P_k} \geq 1$$

$$n(1-y_{2ij}) + \sum_{P_k : l_j \in P_k} I_{P_k} - \sum_{P_k : l_i \in P_k} I_{P_k} \geq 1$$

$$n(1-y_{3ij}) + \sum_{P_k : l_i \in P_k or l_j \in P_k} I_{P_k} - 4 \sum_{P_k : l_i \in P_k, l_j \in P_k} I_{P_k} \geq 0$$

$$y_{1ij} + y_{2ij} + y_{3ij} = 1$$

where n is number of links inside the network ($n = |E|$).

An important point about binary optimization in Eq.15 is that the number of constraints is a polynomial function of n and it is of order $O(n^2)$.

A feasible solution to binary integer programming in Eq. (15) means that graph $N(V, E)$ is identifiable and by solving the optimization problem, we can find the optimum paths which can be used to find delay of each link inside the network.

IV. Conclusion

This work presented a novel approach to estimate links delay in a network. Using the idea of binary compressed sensing, which has received significant attention in the past few years, we provided upper bound on delay recovery of the links inside the network using an end-to-end probe sending method. Sending probes between nodes on the boundary of a network comes with the cost of increasing traffic inside the network. Thus, we showed how to design routing matrix of a given network to minimize number of injected probes while the network remains 1-identifiable.

Our work here is limited to networks which are 1-identifiable. Although it is the simplest identifiability problem as we showed here it is challenging enough to be worth studying. A possible future research avenue is to extend the work here for networks which are $k > 1$ identifiable.

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